# Machine Design
# Strength of Materials
# Statics/Dynamics

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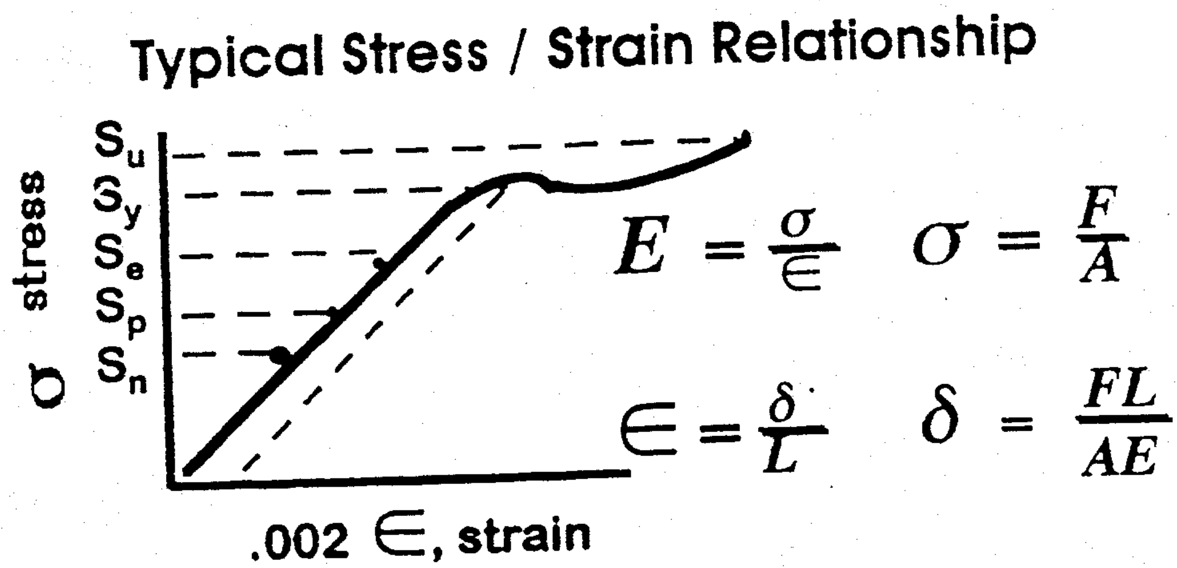
# Content

# Objective

This chapter list equations for normal stress including Hooke's Law and thermal expansion. You will have the equations as well as examples for you to practice.

## Tensile and Compressive Stress and Strain

Typical Stress-Strain Relationship for Engineering Materials




Slope of stress-strain curve in elastic region is the elastic modulus, E

(1)

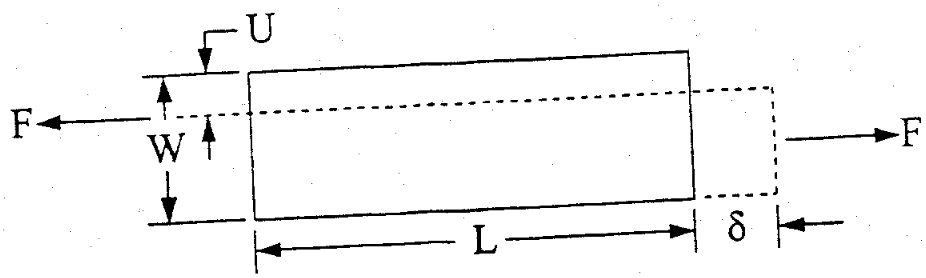
$$\boxed{E = \frac{\sigma}{e}} \qquad e = \frac{\sigma}{E}$$

$$\text{Uniform stress} = \frac{\text{Applied load}}{\text{Resisting Area}} = \boxed{\frac{F}{A} = \sigma} \qquad (2)$$

$$\text{Unit strain} = \frac{\text{Elongation (total)}}{\text{Original length}} = \boxed{\frac{\delta}{L} = e} \qquad (3)$$

Rearranging and combining:

$$\delta = eL = \frac{\sigma}{E}L = \quad \frac{FL}{AE} = \delta \qquad (4)$$




Poisson's ratio, μ is ratio of lateral to axial strain, or

$$\mu = \frac{\frac{U}{W}}{\frac{\delta}{L}}$$ and is typically in the range of 0.3-0.35 for most engineering materials

Allowable stress = max design stress

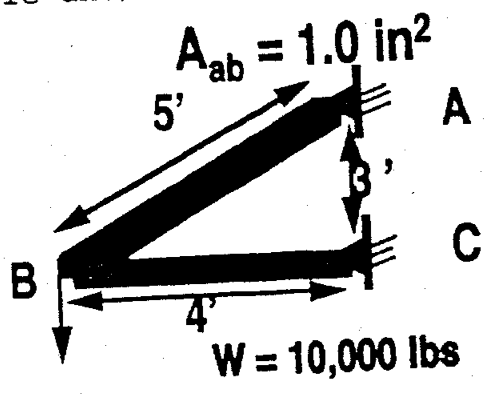
$$(5) \qquad \boxed{\sigma_a = \frac{\sigma y}{S.F.} = \frac{Sy}{S.F.}}$$ S.F. = safety factor

$\sigma_{af} = .6\,\sigma_{yf}$

$\sigma_{af}$ = allowable fatigue stress
$\sigma_{yf}$ = minimum material fatigue stress
S.F. = 1.67 to 3.0 is typical.

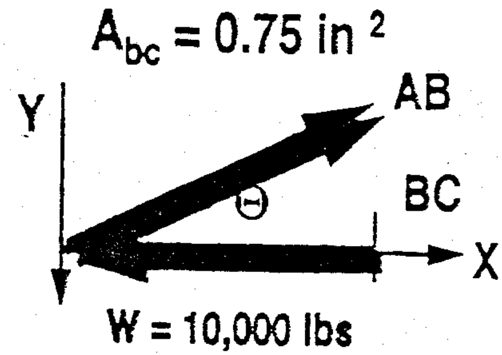
Problem

Find tensile and/or compressive stress in AB and BC.




Area $_{AB}$ = 1in$^2$
Area $_{BC}$ = 0.75 in$^2$

Apply method of joints (statics) at B




$\Sigma Fy = 0$ $\quad\quad$ $AB \sin\theta - 10,000 = 0$

$AB \frac{3}{5} = 10,000$

$AB = 16,667$ pounds

$\Sigma Fx = 0$ $\quad\quad$ $AB \cos\theta - BC = 0$

$16,667 \left(\frac{4}{5}\right) = BC$

$BC = 13,334$ pounds

From (2) $\sigma_{AB} = \frac{F}{A} = \frac{16,667}{1} = 16,667\ psi\ tensile$

$$\sigma_{BC} = \frac{F}{A} = \frac{13,334}{0.75} = 17,778\ psi\ compression$$

## Thermal Growth / Stress

Dimensions of an object change with temperature in proportion to the original length and the temperature change.

$\propto$ = coefficient of thermal expansion

$$\propto = \frac{\Delta L}{L_o(\Delta T)}$$

Thermal "strain" is ratio of the change in length to the original length.

$$e_{Thermal} = \frac{\Delta L}{L_o} = \alpha \Delta T$$

Thermal "stress" is therefore:

$$\sigma_{Thermal} = E \, e_{Thermal}$$

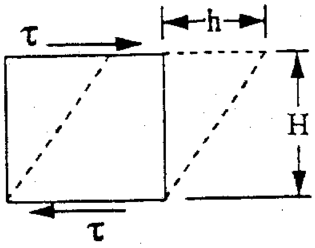
(6) $$\boxed{\sigma_T = E \alpha \Delta T}$$

# Objective

This chapter lists equations for shearing stresses and bending stresses including transverse shear, torsional shear and torsional energy, simple moment-shear diagrams, and bending stress.

Note: stress is induced only if free thermal expansion is prevented.

## Shear Stress and Strain




*Shear Deformation* = $h$

*Unit Shear Deformation* = $\frac{h}{H} = \gamma$

(7) $\boxed{\tau = G\gamma}$ similar to $\sigma = E\,\epsilon$

$\tau$ = shear stress
$\gamma$ = shear strain
$G$ = Modulus of rigidity or shear modulus

(8) $\boxed{G = \frac{E}{2(1+\mu)}}$ $\mu$ = Poisson's Ratio
if $\mu$ = 0.3

$$G = \frac{E}{2.6} = 0.385E$$

MECHANICS of MATERIALS

* Shear stress due to Torsion

(9) $\boxed{\tau = \frac{Tr}{J}}$

T = torque (in #)
r = radius (in)
J = polar moment of inertia

elastic torsion formula

(9.5) $\boxed{\tau = \frac{\theta G D}{2L}}$

D = diameter (in)
L = length (in)
G = shear modulus
θ = angle of twist (radians)

(10) $\boxed{\theta = \frac{TL}{JG}}$

2π Radians = 360°
1 Radian = 57.3°

$$J = \frac{\pi D^4}{32} \quad \textit{for a solid round bar}$$

or $$J = \frac{\pi r^4}{2}$$

(11) $\boxed{E = \frac{1}{2} T \theta}$

<u>Problem:</u> Composite shaft is made of solid steel rod on which is shrunk an aluminum tube. If shaft is 5' long and carries a torque of 15,000 in-#, determine:

a) total angle of twist of the shaft
b) max shearing stress in the steel and aluminum.

Dia of steel rod is 1.5 in
Outside dia of aluminum tube is 2.25 in
$G_{steel} = 12 \times 10^6$ , $G_{alum} = 3.5 \times 10^6$

$$J_s = \frac{\pi D^4}{32} = \frac{\pi}{32}(1.5)^4 = 0.497\ in^4$$

$$J_a = \frac{\pi}{32}(D_o^4 - D_i^4) = \frac{\pi}{32}((2.25)^4 - (1.5)^4) = 2.02\ in^4$$

* θ angle of twist is same for steel and aluminum while total torque is sum of $T_a$ and $T_s$

From (10) $\theta_a = \frac{T_a L}{J_a G_a}$ $\quad \theta_s = \frac{T_s L}{J_s G_s}$

$$T = T_a + T_s = \frac{\theta J_a G_a}{L} + \frac{\theta J_s G_s}{L}$$

$$\theta = \frac{TL}{J_a G_a + J_s G_s} = \frac{15,000(5)(12)}{(2.02)(3.5 \times 10^6) + (0.497)(12 \times 10^6)}$$

$\theta$ = 0.07 radians

From (9.5) $\tau_a = \frac{T_a D_o}{2J_a} = \frac{\theta G_a D_o}{2L} = \frac{0.07(3.5 \times 10^6)(2.25)}{2(60)}$

$\tau_a = 4,590$psi

$$\tau_s = \frac{\theta G_s D_I}{2L} = \frac{0.07(12 \times 10^6)(1.5)}{2(60)} = 10,500\text{psi}$$

<u>Problem</u>

A solid 1.5" dia shaft is made of a ductile material with an elastic shear max stress of 12,000 psi. If shaft is rotating at 500 rpm, what is the maximum horsepower the shaft can transmit if failure is based on initiation of inelastic behavior? Use a factor of safety of 1.5.

$T = \frac{\tau J}{r}$ from (9) $J = \frac{\pi}{2} r^4$ solid shaft

$$T = \frac{(12,000)(\pi/2)(1.5/2)^4}{(1.5/2)}$$

$T = 7948$ in-#

Allowable torque is

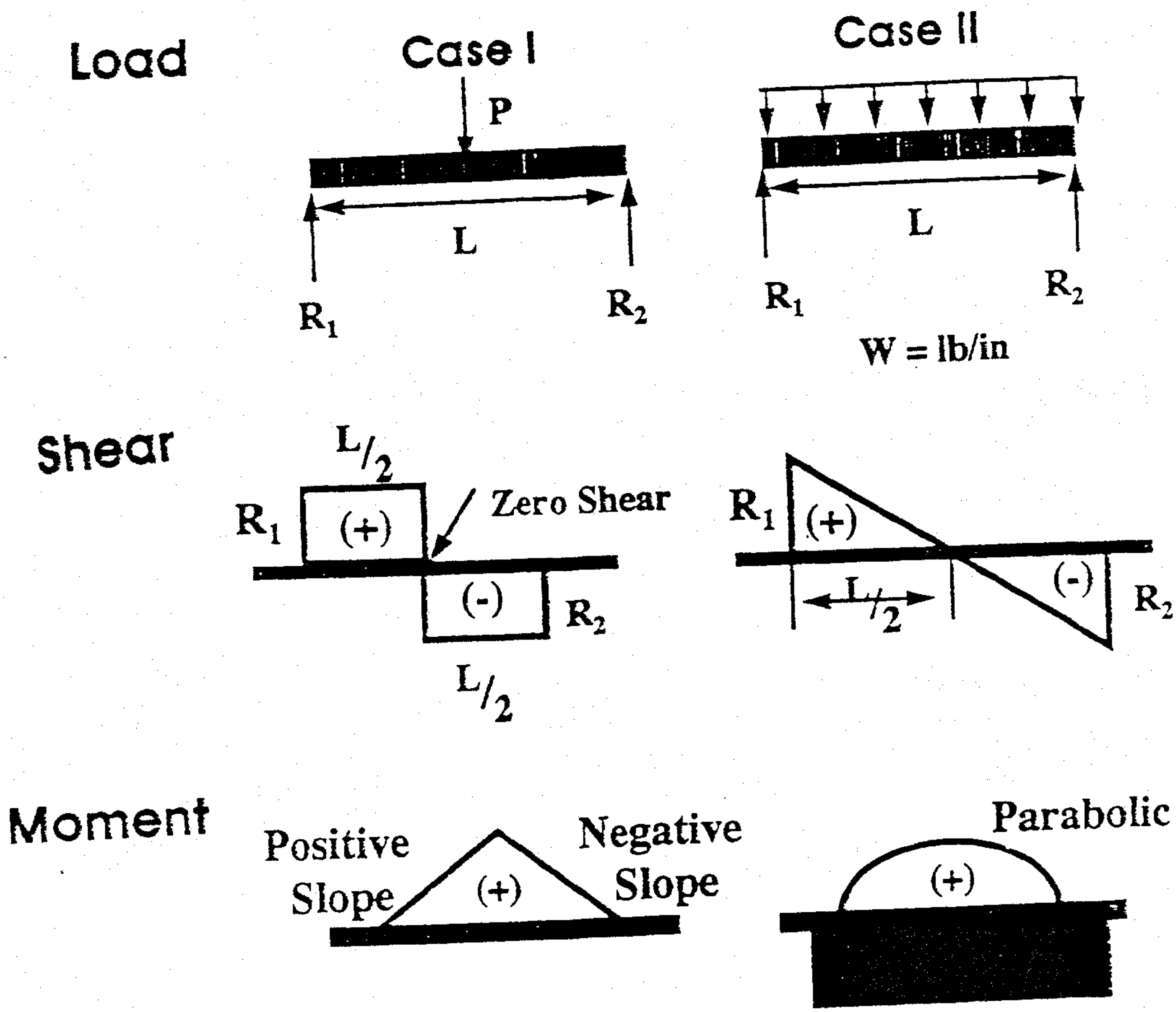
$T_a = \frac{T}{F.S.} = \frac{7948}{1.5} = 5298$ in-#

Power = torque x angular velocity
Power = 5298 in-# x 500 rev/min x $2\pi$ rad/rev.
Power = $1.664 \times 10^7$ in-#/min
1 HP = 33,000 ft #/min = 12 (33,000) in-#/min

$$Max\ HP = \frac{1.664 x 10^7 \ in \cdot \#/\min}{(12)(33,000)\ in \cdot \#/\min /HP}$$

## Shear and Moment Diagrams

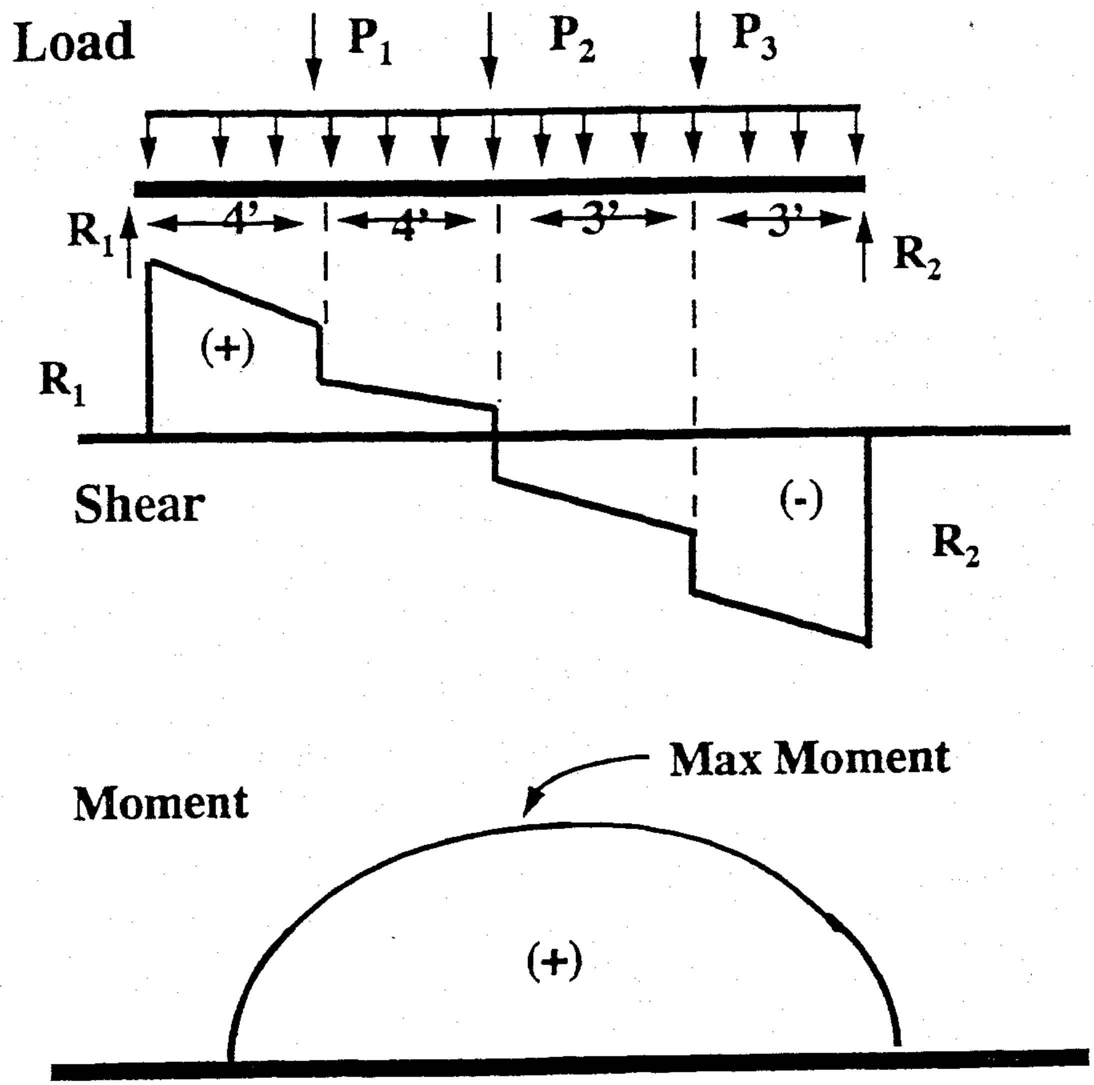
Shear and moment diagrams are graphical representations of the magnitude and sense of shear and moment existing at every point along the length of a beam.




Notes:
1. Max moment is located at point of zero shear
2. Max moment is equal to area under the shear diagram.

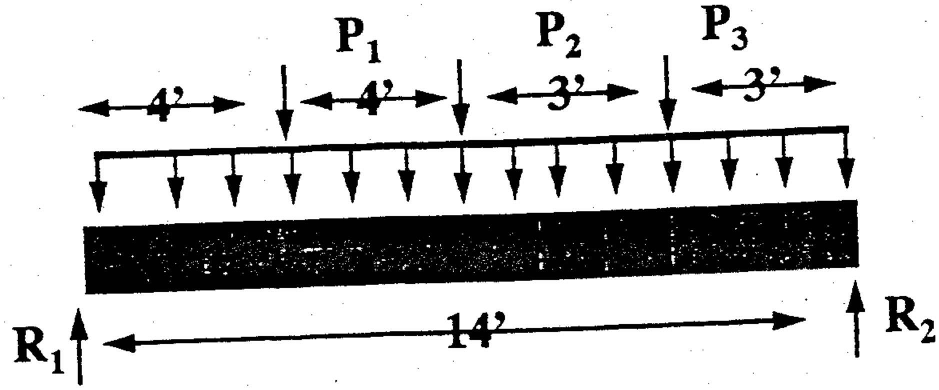
<u>Case I</u> Moment = $R_1$ (L/2)
<u>Case II</u> Moment = 1/2 $R_1$ (L/2)

Case III



Problem

In the shear and moment diagram CASE III the beam is made of yellow pine with an allowable stress of 1000 psi. $P_1$, $P_2$ and $P_3$ are 2000', 4000', and 5500' and respectively. The uniform load is 100'/ft. Find: height of the beam if the width is 12".

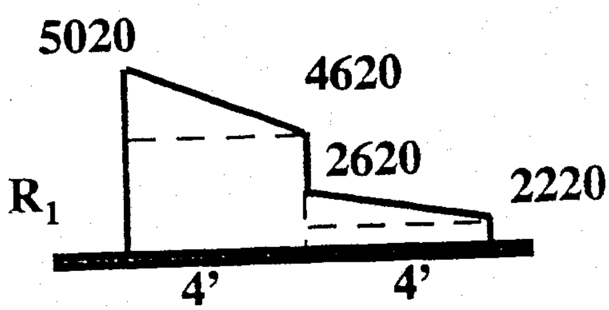



$\Sigma m_{R1} = 0 \; P_1(4) + P_2(8) + P_3(11) + WL(7) - R_2(14) = 0$

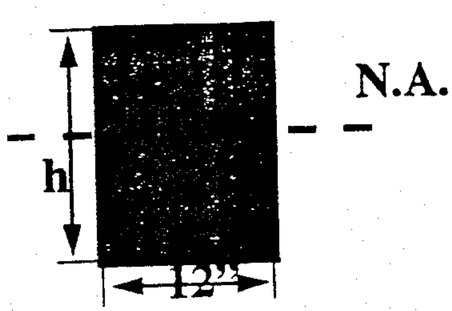
$= 2000 + 4000 + 1\,5500 + 100(14) - 7878 = R_1$

$R_1 = 5020$

From Case III shear diagram (area = moment)




M = (4) (4620) + 1/2 (4) (400) + (4) (2220) + 1/2 (4) (400)
M = 28,960 ft -# = 347,520 in-#




$\sigma_a = \frac{Mc}{I} = \frac{M}{I/c} = \frac{M}{Z}$

$Z = \frac{M}{\sigma_a} = \frac{28,960(12)}{1000}$

$$Z = \frac{I}{c}$$ $$Z = 347\ in^3$$

$$I = \frac{bh^3}{12} \qquad c = \frac{h}{z}$$

$$Z = \frac{bh^3 2}{12(h)} = \frac{bh^2}{6}$$

$$h^2 = \frac{6Z}{b} = \frac{6(347)}{12} = 173.5$$

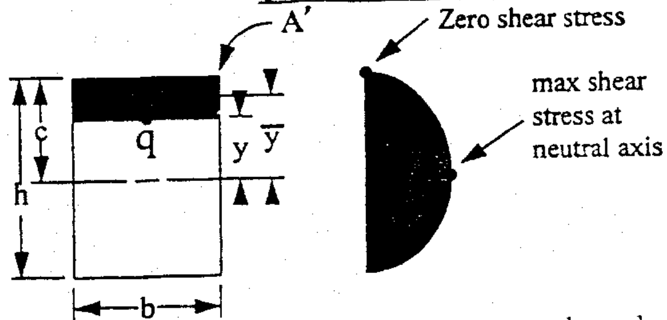
$$h = 13.17\,in \quad or \approx 14''$$

when h = 14"

$$\sigma_{act} = \frac{mc}{I} = \frac{28,960(12)(7)}{2744} = 886\ psi$$

added $S.F. = \frac{\sigma allow}{\sigma act} = \frac{1000}{886} = 1.13$

## Shear Stress in a Beam




Shear stress at any point q may be determined by:

(14) $$\boxed{\tau = \frac{VA^1\bar{y}}{Ib}}$$

V = shear load (shear diagram)
A' = Area outside of point of interest
$\bar{y}$ = distance from neutral axis to centroid of "statical moment" $A^1$

$Q = A^1\bar{y}$ or the

6 = width of section thru q
I = moment of interia of entire cross section

(15) $$\boxed{\tau_{max} = \frac{3V}{2A} = \frac{3V}{2bh}}$$ for rectangle beams

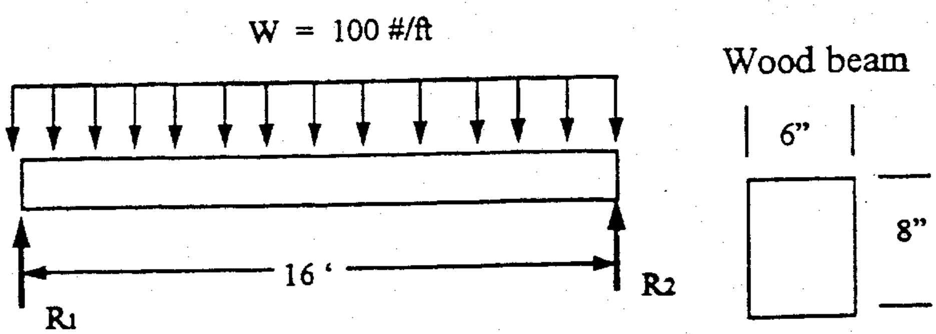
$$\tau_{max} = \frac{4V}{3A} = \frac{4V}{3\pi r^2}$$ for round beams

$$\tau_{max} = \frac{2V}{A} \quad \text{for hollow cylinder}$$

$$\tau_{max} = \frac{V}{A_w} \quad \text{approximate for I beam}$$
Aw = web area

Problem

Find max shear stress

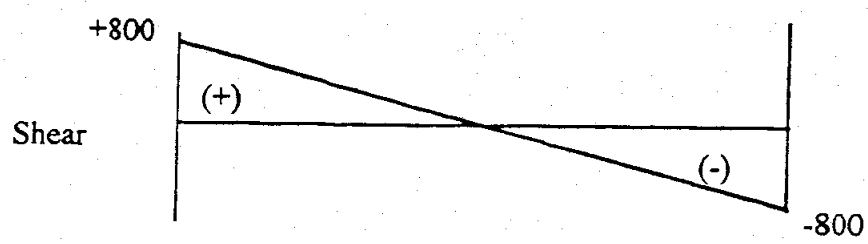



$$\Sigma m_{R1} = 0 = 100(16)(8) - R_2(16) = 0$$

$\Sigma Fy = 0 \qquad WL - R_1 - R_2 = 0$

$100\ (16) - R_1 - 800 = 0$

$R_1 = 800^{\#}$




From (15) $\tau_{max} = \frac{3V}{2A}$ for rectangle

$$\tau_{max} = \frac{3(800)}{2(6)(8)} = 25\,psi$$

The equilibrium equation for horizontal forces for $ABCD$ is then

$$\tau b\, dx + \int_{v_1}^{c_1} \frac{Mv\, dA}{I} = \int_{v_1}^{c_1} \frac{(M + dM)v\, dA}{I} \tag{c}$$

or

$$\tau = \frac{1}{b} \int_{v_1}^{c_1} \frac{dM}{dx} \frac{v\, dA}{I} = \frac{V}{Ib} \int_{v_1}^{c_1} v\, dA \tag{24}$$

In the last form of Eq. (24), shear $V$ has been substituted for $dM/dx$. In Eq. (24), $v\, dA$ represents the moment of the area of the element about the neutral axis. This is integrated over the entire surface from $v_1$, the location where the shearing stress $\tau$ is desired, to the outer edge. This integral can also be written $\bar{v}A_a$, where $A_a$ is the shaded area of view $A$-$A$, and $\bar{v}$ is the distance from its center of gravity to the neutral axis. Equation (24) can then be written

$$\tau = \frac{V}{Ib} \bar{v} A_a \tag{25}$$

As we proved in the preceding section, the shearing stress on the vertical end surfaces at distance $v_1$ from the neutral axis is also equal to the horizontal shear stress $\tau$ as determined by Eq. (24) or (25).

For composite cross sections it is convenient to divide area $A_a$ into several parts, find $\bar{v}A_a$ for each of them, and then add together for the final result. For such beams, Eq. (25) is written

$$\tau = \frac{V}{Ib} \sum \bar{v} A_a \tag{26}$$

The total shear force on the cross section is represented by $V$. The distance from the neutral axis to the point where the shearing stress is desired is given by $v_1$.

*Example 12.* Find the transverse shear in the material 3 in. from the top surface for the beam of Fig. 1-21(a).

*Solution.* As is shown in Example 3, the center of gravity of the cross section in Fig. 1-21(b) is found to be 2 in. up from the bottom. As shown in Example 6, the moment of inertia about the horizontal axis through the center of gravity is found to be 33.33 in.$^4$

Referring to Fig. 1-21(c), it is seen for location 3 in. from the top that $\bar{v} = 2.5$ in. and $A_a = 3$ in.$^2$. Substitution in Eq. (25) gives

$$\tau = \frac{10{,}000}{33.33 \times 1} \times 2.5 \times 3 = 2{,}250 \text{ psi}$$

It is of course immaterial whether $A_a$ is taken above or below the location at which the stress is desired. Equation (26) gives

$$\tau = \frac{10{,}000}{33.33 \times 1}(1.5 \times 4 + 0.5 \times 3) = 2{,}250 \text{ psi}$$

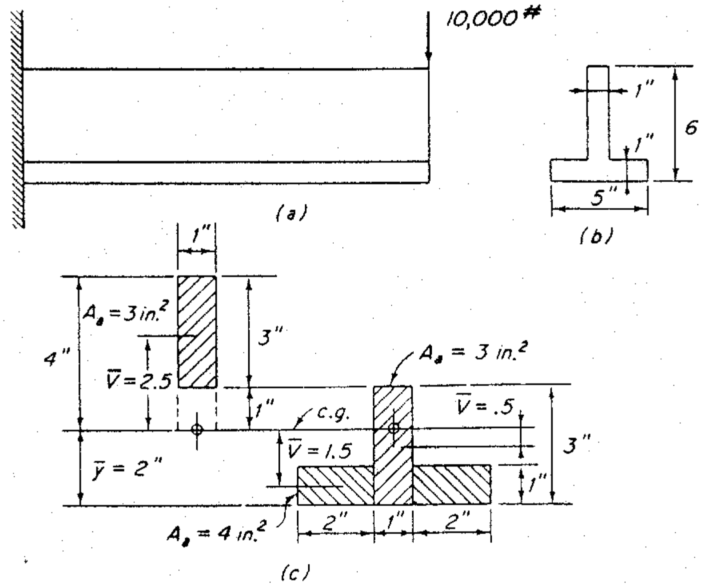



FIGURE 1-21 Examples 3, 6, and 12.

*Example 12A.* Find the transverse shear stress in the material 75 mm from the top surface for the beam of Fig. 1-21(a); but use the dimension given for Example 3A. The force at the end of the beam is 45,000 N.

*Solution.* A large-size sketch should be made and dimensioned for the cross section in mm. As found in previous examples, $\bar{y} = 50$ mm and $I = 13{,}021{,}000$ mm⁴. When the area above 75 mm is considered:

$$\bar{v} = 62.5 \text{ mm}, \qquad A_a = 1{,}875 \text{ mm}^2$$

By Eq. (25): $\tau = \dfrac{V}{Ib}\bar{v}A_a = \dfrac{45{,}000}{13{,}021{,}000 \times 25} \times 62.5 \times 1{,}875 = 16.2 \text{ MPa}$

When the areas below 75 mm are considered:

Let the vertical stem extend to the bottom, $\bar{v} = 12.5$ mm and $A_a = 1{,}875$ mm².
For the remainder of the flange, $\bar{v} = 37.5$ mm and $A_a = 2{,}500$ mm².

By Eq. (26): $\tau = \dfrac{45{,}000}{13{,}021{,}000 \times 25}(12.5 \times 1{,}875 + 37.5 \times 2{,}500)$

$= 16.2$ MPa

When Eq. (24) is applied to rectangular cross section, $dA = b\,dv$ and $c_1 = h/2$. After making these substitutions and integrating, the following result is obtained.

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - v_1^2\right) = \frac{3V}{2A}\left(1 - \frac{4v_1^2}{h^2}\right) \tag{27}$$

## BEAM DEFLECTIONS

Beam deflections may be determined by several different methods including:

1. double integration method
2. moment area method
3. strain energy method
4. look it up in a table method
5. superposition

The double integration method relates deflection and slope to the beam radius of currature. If

- y = deflection

$$y^{1} = \frac{dy}{dx} = \text{slope}$$

$$y^{11} = \frac{d^2y}{dx^2} = \frac{M}{EI} \quad \text{(moment)}$$

$$y^{111} = \frac{d^3y}{dx^3} = \frac{V}{EI} \quad \text{(shear)}$$

Then the deflection at any point is:

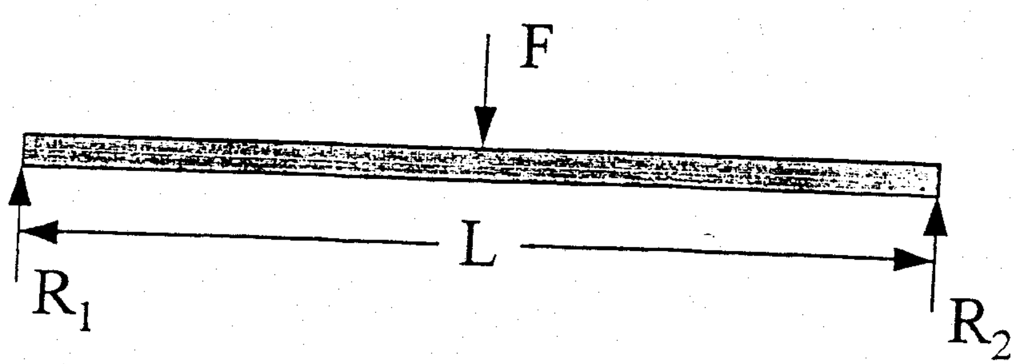
$$y = \frac{1}{EI} \int [\int m\ (x)\ dx]$$

M (x) is the moment function.

Beam deflections may also be found from tables of beam formulas located in most engineering handbooks. The method of superposition may be used to evaluate complex load cases by combining two or more simpler formulas. The deflections at a point due to the individual loads may be added to find the total deflection.

Some typical deflections are:

1. Simple supports with center load



From (12)

$$I_T = \Sigma I_o + \Sigma A d^2$$

$$I_T = 287\ in^4$$

From (11)

<u>Tensile</u> (steel) (strongest)*

$$\sigma_T = \frac{McN}{I} =$$

$$\sigma_T = \frac{5000(12)(1.5)(20)}{287}$$

$\sigma_T =$ 6260 psi

<u>Compression</u> (wood) (weakest)**

$$\sigma c = \frac{Mc}{I} =$$

$$\sigma_c = \frac{5000(12)(7-1.5)}{287}$$

$$\sigma_c = 1150\ psi$$

$$* \ \sigma_{strongest} = \frac{NMc_{strongest}}{I_{centroid}}$$

$$** \ \sigma_{weakest} = \frac{MC_{weakest}}{I_{centroid}}$$

From (13)

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{60(.5)+24(4)}{84}$$

$$\bar{y} = 1.5''$$

$$I_w = \frac{bh^3}{12} = \frac{4(6)^3}{12} = 72\ in^4$$

$$I_{s-w} = \frac{bh^3}{12} = \frac{60(1)^3}{12} = 5\ in^4$$

$$A_1 d_1 = (24)(4-1.5)^2 = 150$$

$$A_2 d_2 = (60)(1.5-5)^2 = 60$$

$M = \frac{PL}{4}$ from tables or shear and moment diagram ft #

$$M = \frac{2000(10)}{4}$$

<u>Safety Factor</u> of wood

$$S.F. = \frac{design\ \ allowable\ \ stress}{actual\ \ stress}$$

$$S.F. = \frac{1500}{1150} = 1.3\ \text{min.}$$

# Objective

This chapter gives an example of stress calculation for composite materials. Please also read the review book and you will be able to solve this type of problem.

# Bending Stresses

Normal Bending Stress

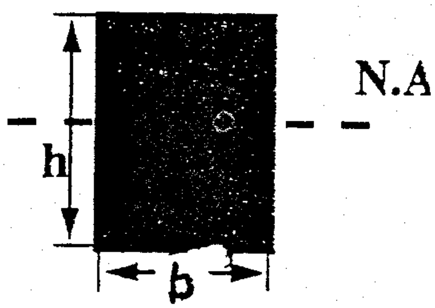
(11) $\boxed{\sigma = \frac{My}{I} = \frac{Mc}{I}}$

M = moment in-#
y = distance from neutral axis (in)
c = max distance from neutral axis. (in)
I = moment of inertia (in$^4$)
(plane perpendicular to moment)

$$\sigma = \frac{M}{I/C} = \frac{M}{Z}$$

$Z = I/c =$ *section modulus*

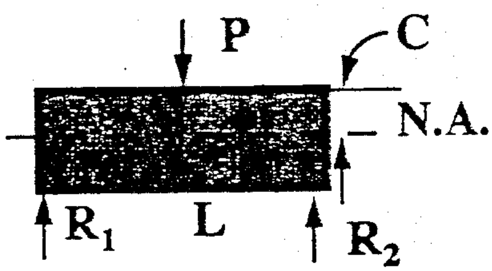
$I = \frac{bh^3}{12}$ *for rectangle cross-section*




(12) $\boxed{I_x = I_{cG} + Ad^2}$ Parallel Axis Theorem

Used to find moment of inertia of a composite figure about its own centroidal axis.

$I_{cG}$ = moment of inertia about neutral axis for each figure within the composite figure.
A = Area of each figure (in$^2$)
d = distance between composite figure centroidal axis and each figures neutral axis. (in)

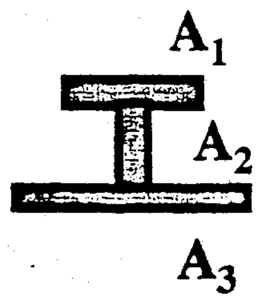



In pure bending, load is distributed across beam as shown

Zero bending stress occurs at the neutral axis. Maximum bending stress occurs at the fiber furthest from N.A. For a beam with a symmetrical cross section, the N.A. occurs at the line of symmetry.

For an unsymmetrical cross section, locate the composite neutral axis by breaking the section down into elements and solving for $\bar{y}$.

(13) $\boxed{\bar{y} = \frac{\Sigma A y}{\Sigma A}} = \frac{A_1 y_1 + A_2 y_2 7 A_3 y_3}{A_1 + A_2 + A_3}$



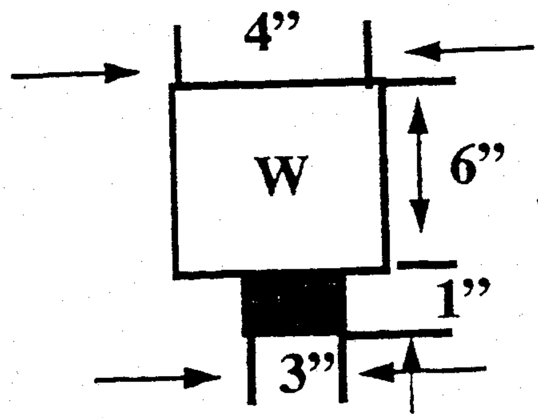

Problem:

Wood beam 4" wide by 6" deep is re-inforced on the tension side by a steel plate 1" thick by 3" wide securely bolted. Beam is simply supported and has a clear span of 10'. It carries a concentrated load of 2000# at the center.

a) Calculate tensile and compressive stresses in the outer most fibers of the composite beam.

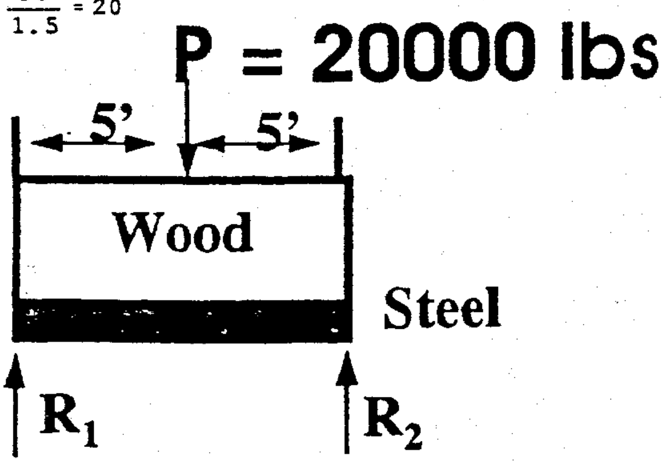
b) If the allowable design stress is 1500 psi, what is the minimum safety factor in the wood beam.

Ew = 1.5 x 106 psi E$_s$ = 30 x 10 psi




weakest

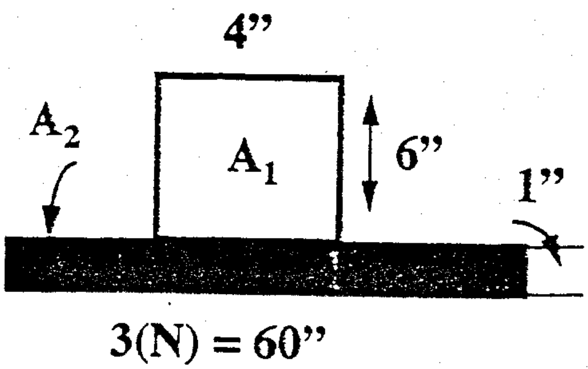
$N = \frac{E_s}{E_w} = \frac{30}{1.5} = 20$



$$\Sigma M_{R_1} = 0$$
$$P(5) - R_2(10) = 0$$
$$P(5) = R_2(10)$$
$$R_2 = 1000\,\#$$

$$\Sigma F_y = 0$$
$$R_1 + R_2 - P = 0$$
$$R_1 + 1000 - 2000 = 0$$
$$R_1 = 1000\,\#$$

Transform beam so that both are wood by using N above (ration of modulus of elasticity)




$A_1 = (6)\ (4) = 24\ in_2$
$A_2 = (1)\ (60) = 60\ in_2$
$A_T = 84\ in_2 = \Sigma\ A$

# Objective

This chapter gives a very good example of buckling of columns. By following the analysis of the flow chart, you will be able to solve most of the problems.

## You Are the Designer

You are a member of a team that is designing a commercial compactor to reduce the volume of cardboard and paper waste so it can be transported easily to a processing plant. Figure 6–1 is a sketch of the compaction ram that is driven by a hydraulic cylinder under several thousand pounds of force. The connecting rod between the hydraulic cylinder and the ram must be designed as a column because it is a relatively long, slender compression member. What shape should the cross section of the connecting rod be? From what material should it be made? How is it to be connected to the ram and to the hydraulic cylinder? What are the final dimensions of the rod to be? You, the designer, must specify all of these factors.

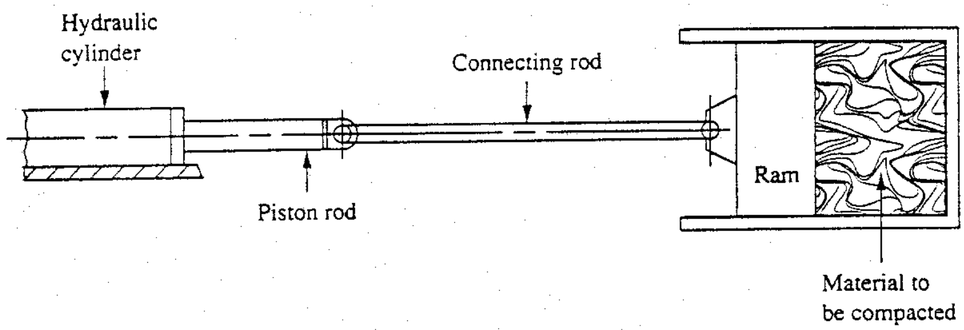



Figure 6–1 Waste Paper Compactor

■ ■ ■

## 6–1 OBJECTIVES OF THIS CHAPTER

A *column* is a structural member that carries an axial compressive load and that tends to fail by elastic instability, or buckling, rather than by crushing the material. *Elastic instability* is the condition of failure in which the shape of the column is insufficiently rigid to hold it straight under load. At the point of buckling, a radical deflection of the axis of the column occurs suddenly. Then, if the load is not reduced, the column will collapse. Obviously this kind of catastrophic failure must be avoided in structures and machine elements.

Columns are relatively long and slender. If a compression member is so short that it does *not* tend to buckle, failure analysis must use the methods presented in Chapter 5. This chapter presents several methods of analyzing and designing columns to ensure safety under a variety of loading conditions.

### Specific Objectives

After completing this chapter, you will be able to:

1. Recognize that relatively long, slender compression members must be analyzed as columns to prevent buckling.

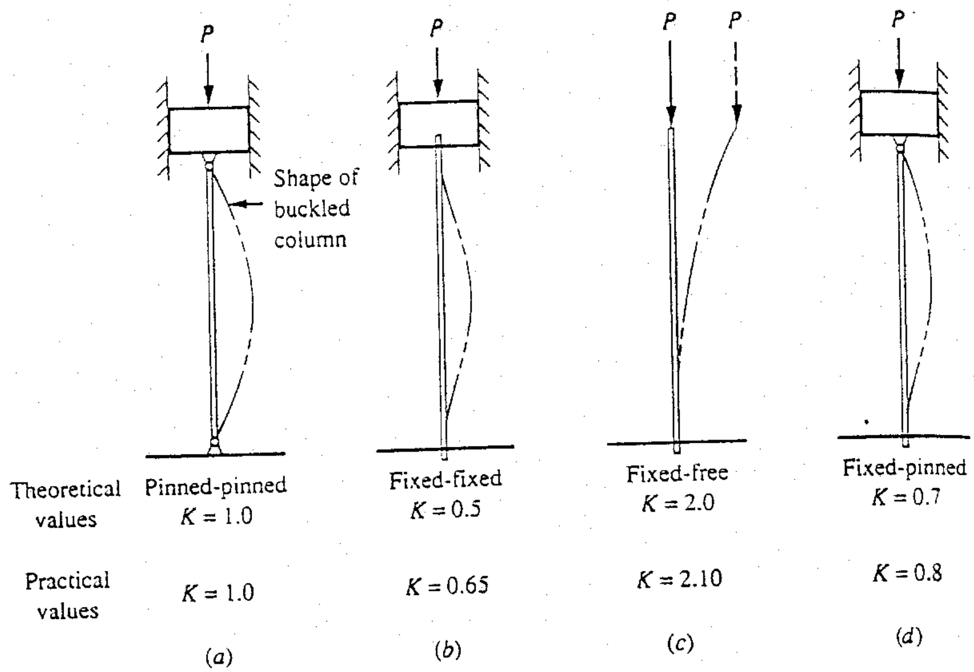



**Figure 6–3 Values of $K$ for Effective Length, $L_e = KL$, for Different End Connections**

A *fixed end* is one that is held against rotation at the support. An example is a cylindrical column inserted into a tight-fitting sleeve that itself is rigidly supported. The sleeve prohibits any tendency for the fixed end of the column to rotate. A column end securely welded to a rigid base plate is also a good approximation of a fixed-end column.

The *free end* can be visualized by the example of a flagpole. The top end of a flagpole is unrestrained and unguided, the worst case for column loading.

The manner of support of both ends of the column affects the *effective length* of the column, defined as

$$L_e = KL \tag{6–2}$$

where $L$ is the actual length of the column between supports and $K$ is a constant dependent on the end fixity, as illustrated in Figure 6–3.

The first values given for $K$ are theoretical values based on the shape of the deflected column. The second values take into account the expected fixity of the column ends in real, practical structures. It is particularly difficult to achieve a true fixed-ended column because of the lack of complete rigidity of the support or the means of attachment. Therefore, the higher value of $K$ is recommended.

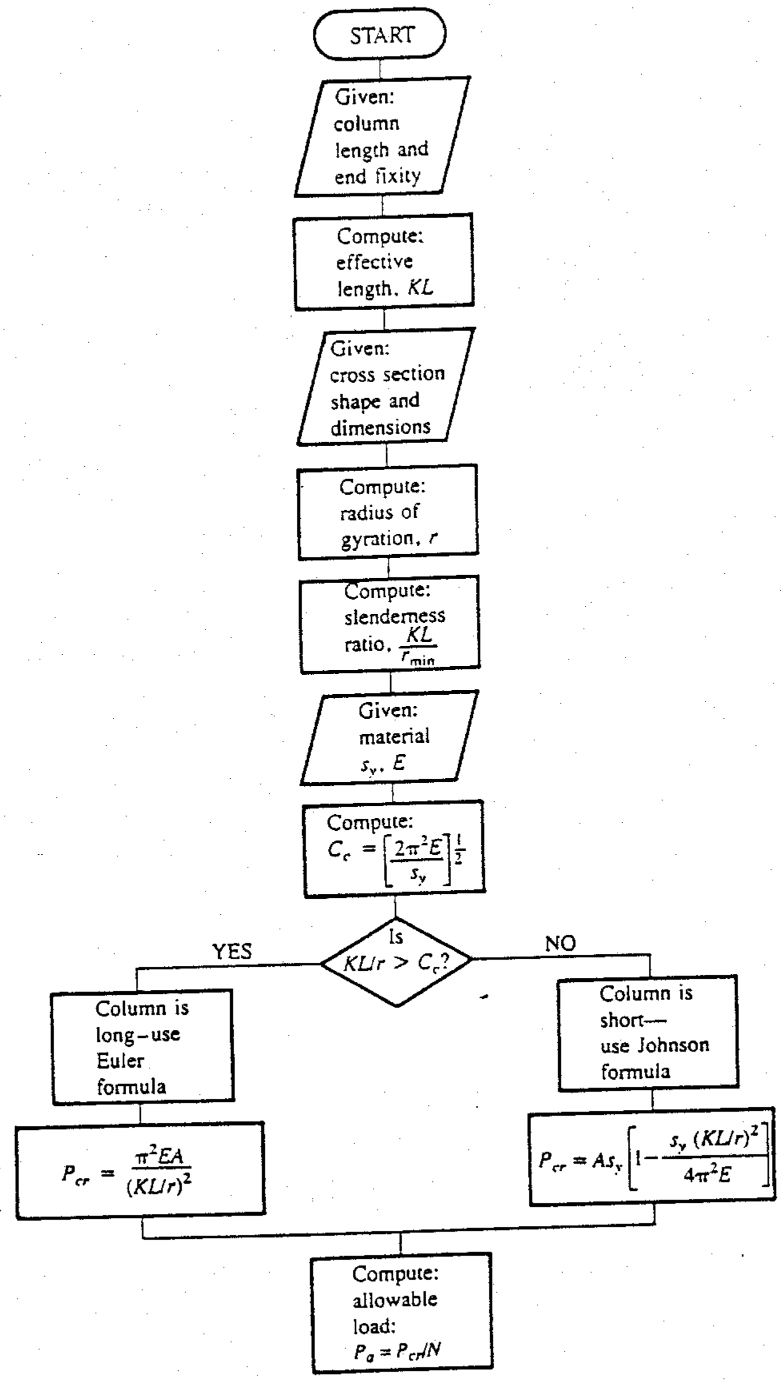
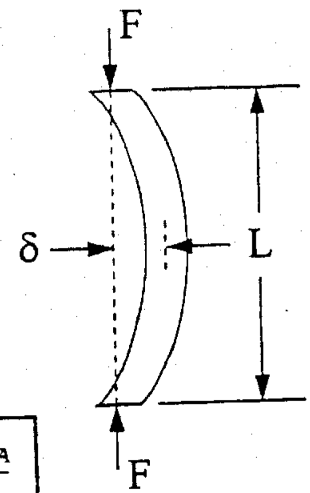



**Figure 6–4 Analysis of a Straight, Centrally Loaded Column**

## Columns

Columns typically fail by buckling after reaching a state of elastic instability under critical load for:




For a long column with pinned ends (free to flex)

k = radius of gyration

$$k = \sqrt{\frac{I}{a}}$$

(16)
$$\boxed{F_{cr} = \frac{\pi^2 E I}{L^2} = \frac{(k\pi)^2 E A}{L^2}}$$

Euler load is theoretial maximum load that an initially straight column can support without buckling.

Column stress is:

(17)
$$\boxed{\sigma_{cr} = \frac{F_{cr}}{A} = \frac{\pi^2 E}{\left(\frac{L}{K}\right)^2}}$$

$\frac{L}{k}$ = slenderness ratio

Assumes the column is long so that the Euler stress is reached before the yield stress is reached.

| | |
|---|---|
| $\frac{L}{K} < 30$ | are stocky "columns" and have little chance of buckling |
| $\frac{L}{k}$ 30-120 | are in the inelastic range and the Euler equations should not be used |
| $\frac{L}{K}$ 120-200 | Euler equations range |

$\frac{L}{k} > 200$ are too thin to be considered a load carrying a column

For cases where both ends are not pinned, the critical load may be expressed as

(18) $$\boxed{F_{cr} = \frac{n\pi^2 EI}{L^2}}$$ where n = end condition coefficient

| End condition | n |
|---|---|
| pinned-pinned | 1.0 |
| fixed-free | 0.25 |
| fixed-fixed | 4.0 |
| fixed-pinned | 2.0 |

Since the above Euler equations assume Hooke's Law applies - short columns may yield <u>before</u> $F_{cr}$ is reached.

In general $F_{cr} \leq \frac{Sy}{2}$ is acceptable

Sy = yield strength in compression.

For intermediate and short columns eccentricities are an important factor so use "secant formula".

(19) $$\boxed{\sigma_{cr} = \frac{F_{cr}}{A} = \frac{Sy}{1 + \frac{ec}{K^2}\sec\Phi}}$$

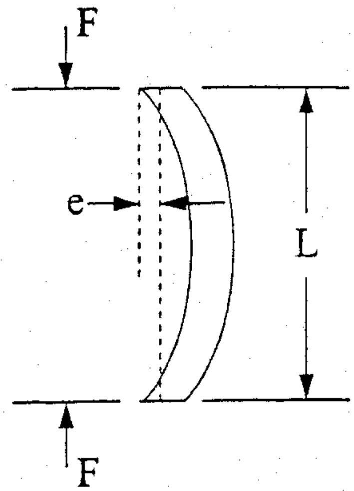
where:

$$\Phi = \frac{1}{2}\left(\frac{L}{k}\right)\sqrt{\frac{F_{cr}}{AE}}$$

and:

$\frac{ec}{k^2}$ = eccentricity ratio

(use 0.25 if not known)




e = eccentricity
c = distanc from neutral axis to outermost fiber

Problem

S - type 7 x 20 A 36 steel I-beam 10 ft long is used as a column. What is working stress for a safety factor of 3?

$E = 30 \times 10^6$
$A = 5.88 \text{ in}^2$

Sy = yield strength = 36,000 psi
$I = 3.17 \text{ in}^4$
$K = 0.734 \text{ in}$

Note: I and K are the smallest I-beam values.

From (17) $\sigma_{cr} = \dfrac{\pi^2 E}{\left(\dfrac{L}{K}\right)^2} = \dfrac{\pi^2 (30 \times 10^6)}{\left(\dfrac{10(12)}{0.734}\right)} = 11{,}078\,psi$

Since 11,078 is less than Sy (36,000 psi) the Euler formula is valid. Since

$\sigma_{cr} = \dfrac{F_{cr}}{A} \le \dfrac{Sy}{2}$ also $\left(11{,}078 \le \dfrac{36{,}000}{2}\right)$

The stress is acceptable to prevent buckling.

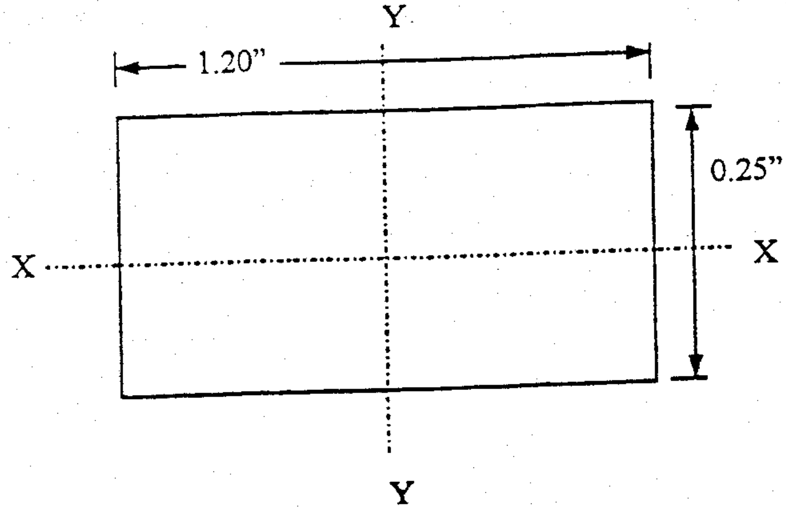
$\dfrac{L}{K} = \dfrac{10(12)}{0.734} = 163$ "Euler range"

$\sigma_{allowable} = \dfrac{\sigma_{cr}}{3} = \dfrac{11{,}078}{3} = 3{,}692\,psi$

working stress

Problem

A structural steel strut with rounded ends (pinned-pinned) has the cross-section shown, and is 10" long. At what compressive axial load will the strut fail? $E = 30 \times 10^6$ psi




First: find slenderness ratio to see if Euler formula is applicable.

$$I_x = \frac{bh^3}{12} = \frac{1.2(0.25)^3}{12} = .00156\ in^4$$

$$I_y = \frac{bh^3}{12} = \frac{0.25(1.2)^3}{12} = 0.36\ in^4$$

$$k = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{.00156}{1.2(0.25)}} = .0721\ in$$

Use smallest I, or $I_x$

$$\frac{L}{k} = \frac{10}{0.0721} = 138 \quad \text{(within Euler range)}$$

From (17)

$$\sigma_{cr} = \frac{\pi^2 E}{(L/k)^2} = \frac{\pi^2 (30 \times 10^6)}{(138)^2} = 15,543\ psi$$

$$\sigma_{cr} = \frac{F_{cr}}{A}$$

$$F_{cr} = \sigma_{cr} A = 15,543(1.2)\ (0.25) = 4663\ lb$$

# Objective

This chapter gives an example of stress calculation for pressure vessels by using the concept of Mohr's circle in 3-dimension.

## EXAMPLE 13.1

A thin-walled cylinder with closed ends for which $r_1 = 0.50$ m and $r_2 = 0.52$ m is subjected to internal pressure $p_1 = 2$ MPa. Determine (a) the absolute maximum shearing stress on the inner surface of the cylinder, (b) the absolute maximum shearing stress on the outer surface of the cylinder, and (c) the normal and shearing stresses in the wall of the cylinder on a plane inclined to the longitudinal axis of the cylinder through a 30° angle.

*SOLUTION*

(a) Consider a three-dimensional stress element on the inner surface of the cylinder such that two of its planes are parallel to the cylinder longitudinal axis and two perpendicular to this axis. The two planes parallel to the longitudinal axis are subjected to the circumferential stress $\sigma_t$ given by Eq. 13.1. Thus

$$\sigma_t = \frac{p_1 r_1}{t} = \frac{2 \times 0.50}{0.02} = 50 \text{ MPa}$$

This stress is tensile and, as mentioned earlier, it is a principal stress. The two planes perpendicular to the longitudinal axis of the cylinder are subjected to the longitudinal stress $\sigma_z$ given by Eq. 13.2. Therefore,

$$\sigma_z = \frac{p_1 r_1}{2t} = 25 \text{ MPa}$$

This stress is also a tensile principal stress. The third set of two parallel planes, one of which is the inner cylindrical surface, is subjected to a radial stress which is equal in magnitude to the applied pressure $p_1 = 2$ MPa. This last stress is obviously a compressive principal stress. Thus the stress element on the inner surface of the cylinder is subjected to the following three principal stresses, which are consistent with the condition that algebraically, $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

$$\sigma_1 = \sigma_t = 50 \text{ MPa}$$

$$\sigma_2 = \sigma_z = 25 \text{ MPa}$$

$$\sigma_3 = \sigma_r = -2 \text{ MPa}$$

Therefore, the absolute maximum shearing stress is given by Eq. 2.16. Thus

$$|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2} = \frac{50 + 2}{2}$$

$$= \boxed{26 \text{ MPa}}$$

This stress acts on a plane that bisects the 90° angle between $\sigma_1$ and $\sigma_3$.

(b) Consider a three-dimensional stress element on the outer surface of the cylinder whose planes are defined in the same manner as those in part (a). The circumferential and longitudinal stresses are identical to those found in part (a). However, the radial stress on the last set of two parallel planes, one of which is the outer cylindrical surface, is zero, since it is a free surface. Thus the three principal stresses on the outer surface of the cylinder are

$$\sigma_1 = \sigma_t = 50 \text{ MPa}$$

$$\sigma_2 = \sigma_z = 25 \text{ MPa}$$

$$\sigma_3 = \sigma_r = 0$$

Hence the absolute maximum shearing stress is given by Eq. 2.16:

$$|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2} = \frac{50 + 0}{2}$$

$$= \boxed{25 \text{ MPa}}$$

This stress acts on a plane that bisects the 90° angle between $\sigma_1$ and $\sigma_3$.

(c) The Mohr's circle for the two-dimensional or plane stress condition defined by $\sigma_1$ and $\sigma_2$ is constructed as shown in Fig. 13.2(a). As shown in the two-dimensional stress element of Fig. 13.2(b), the plane of interest $B$–$B$ is inclined to the plane of $\sigma_1$ through a 30° angle shown clockwise in the sketch, although a counterclockwise angle would serve the same purpose. This plane is represented on the Mohr's circle by point $B$, whose coordinates give the values of the normal and shearing stresses on this plane. Thus

$$\sigma_B = OC + R \cos 60 = \frac{50 + 25}{2} + \left(\frac{50 - 25}{2}\right) \cos 60$$

$$= 37.50 + 6.25$$

$$= \boxed{43.75 \text{ MPa}}$$

$$\tau_B = R \sin 60$$

$$= \boxed{10.83 \text{ MPa}}$$

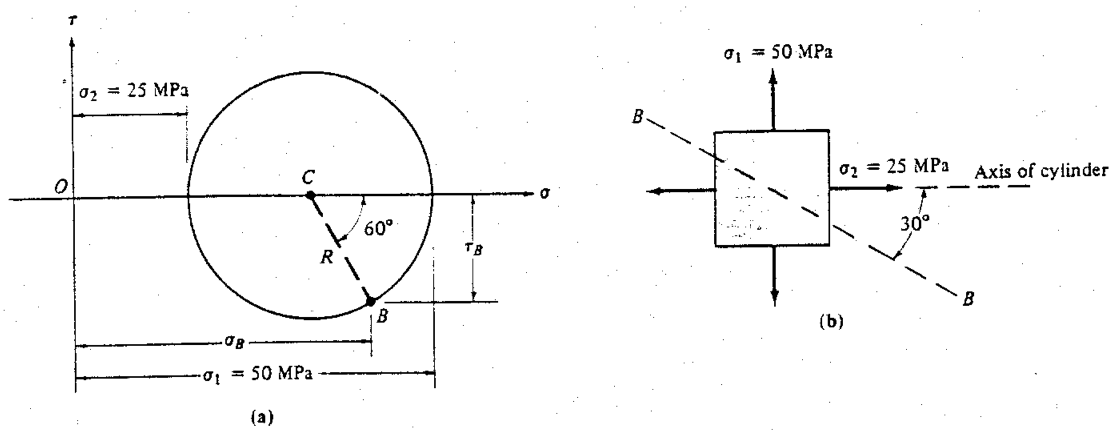
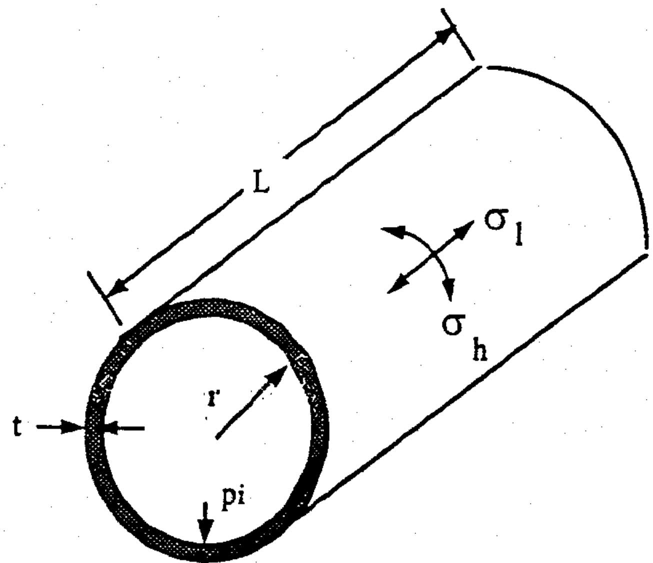



**FIGURE 13.2**

## PRESSURE VESSELS AND THIN WALLED CYLINDERS




A cylinder can be considered "thin-walled" if its thickness to diameter ratio is less than 0.1.

$\sigma_h$ = hoop (tangential) stress
$\sigma_l$ = longitudinal (axial) stress
$\sigma_h = 2\sigma_l$

$$\sigma_h = \frac{F}{A} = \frac{(P_i - P_o)\,rL}{tL} = \frac{(P_i - P_o)\,r}{t}$$

when $P_i$ = psig; $P_o = 0$ or

(20) $\boxed{\sigma_h = \frac{P_i r}{t} = \frac{P_i D}{2t}}$ Hoop Stress

$$\sigma_l = \frac{F}{A} = \frac{(P_i - P_o)\,\pi r^2}{2\pi r t} = \frac{(P_i - P_o)\,r}{2t}$$

When $P_i$ = psig; $P_o = 0$

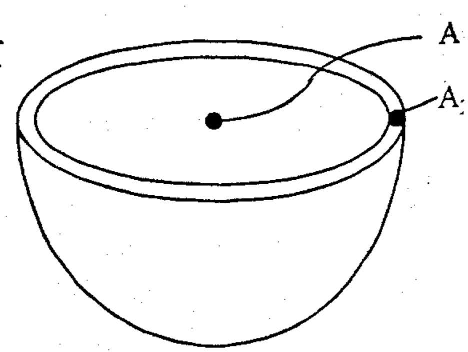
(21) $\boxed{\sigma_l = \frac{P_i r}{2t} = \frac{P_i D}{4t}}$ Longitudinal Streess

For a sphere $\sigma_h = \sigma_l = \frac{Pr}{2t}$

Problem

A sphere contains air pressure of 15000 psig and measures 72" O.D. and 69" ID. What is hoop and axial stress?

hopp stress = longitudinal stress = $\frac{Pr}{2t}$




From (21) $\sigma_h = \frac{Pr}{2t} = \frac{1500(34.5)}{2(1.5)} = 17{,}520psi$

Using force and area method we get:

$$F = PA_1 = 1500\left(\frac{\pi}{4}D_1^2\right) = 1500\left(\frac{\pi}{4}(69)^2\right)$$

$$F = 5{,}608{,}934\ 16$$

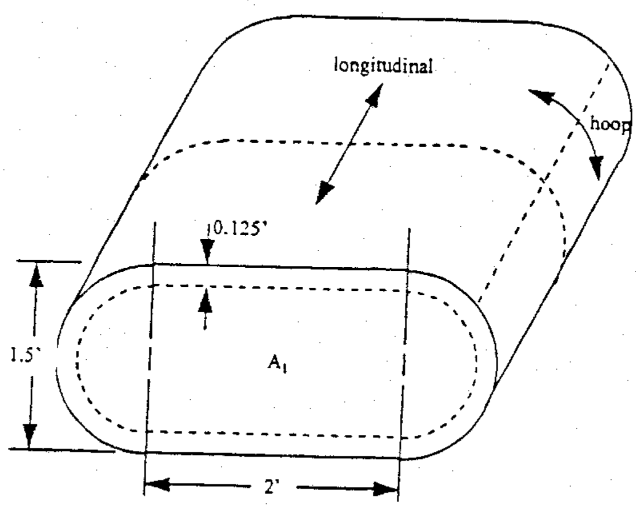
$$S = \frac{F}{A_2} = \frac{5{,}608{,}934}{\frac{\pi}{4}D_2^2 - \frac{\pi}{4}D_1^2} = \frac{5{,}608{,}934}{\frac{\pi}{4}(72)^2 - \frac{\pi}{4}(69)^2} = \frac{5{,}608{,}934}{332.22}$$

$$S = 16{,}883\,psi$$

* $\dfrac{Pr}{2t}$ is more conservative method.

Problem

Fuel tank has internal pressure of 125 psig. Thickness is 1/8" steel place. Find hoop and longitudinal stress.




D = 2' + 1.5' = 3.5' = 42"

Hoop sress

From (20) $\sigma_h = \dfrac{Pr}{t} = \dfrac{PD}{2t} = \dfrac{125(43)}{2(0.125)} = 21{,}000\,psi$

Longitudinal Stress

$F$ = *force acting on area* $A_1$

# Objective

This chapter gives examples for the calculation for thick-wall cylinders.

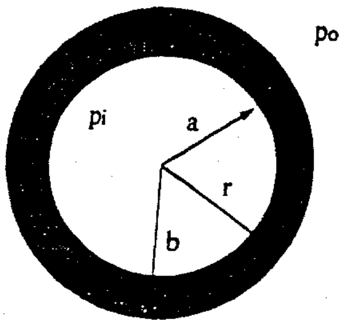
$$F = PA_1 = 125\left[(2)(1.5) + \frac{\pi}{4}(1.5)^2\right]144$$

$$F = 85{,}808\ lb$$

$$\sigma_l = \frac{F}{A} = \frac{85{,}808}{2Lt + \pi Dt} = \frac{85{,}808}{2(24)(0.125) + \pi(18)(0.125)}$$

$$\sigma_l = 6{,}566\ psi$$

## THICK WALLED CYLINDERS




a = inside radius
b = outside radius
r = any radius between a and B
$P_o$ = outside pressure
$P_i$ = inside pressure

General case to find stress at any distance r.

Tangential stress:

$$\sigma_t = \frac{P_i a^2 - P_o b^2 + \frac{a^2b^2(P_o - P_i)}{r^2}}{b^2 - a^2}$$

Radial stress:

$$\sigma_r = P_i a^2 - P_o b^2 =+ \frac{\frac{a^2b^2(P_o - P_i)}{r^2}}{b^2 - a^2}$$

positive stress = tension
negative stress = compression

when using external pressure $P_o = 0$

$$\sigma_t = \frac{a^2 P_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

with $P_o$ = 0 the max stress is at inner wall where r = a 31

max stress with $P_o$ = 0

$$(22)\quad \sigma_t = P_i\left(\frac{b^2 + a^2}{b^2 - a^2}\right) \text{ or } P_i\left(\frac{1 + (a/b)^2}{1 - (a/b)^2}\right)$$

$$\sigma_r = -P_i$$

when $P_i$ = 0 the stresses on the outer surface are:

$$(23)\quad \sigma_t = -P_o\left(\frac{b^2 + a^2}{b^2 - a^2}\right) \text{ or } -P_0\left(\frac{1 + (a/b)^2}{1 - (a/b)^2}\right)$$

$$\sigma_r = -P_o$$

A longitudinal stress due to internal pressure acting against end platis can be calculabed by:

$$(24)\quad \boxed{\sigma_L = \frac{P_i a^2}{b + a)\, t}}$$

t = thickness of wall.

## Strain

Both diametrical strain Δ D/D and circumferential strain Δ C/C are equal in a circular cylinder under pressure loads.

$$(25)\quad \boxed{\Delta D/D = \Delta c/c = \frac{\sigma t - \mu\,(\sigma_r + \sigma_L)}{E}}$$

μ = Poisson's Ratio
E = modulus of elasticity

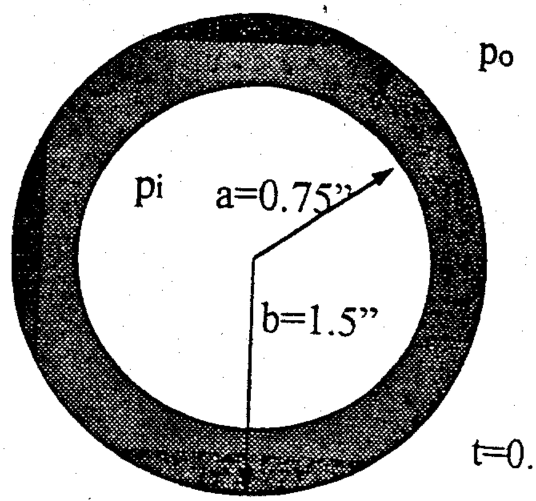
## Problem

A steel cylinder of 1.5" ID and 3" OD is pressurized internally to 12,000 psi. a) the cylinder has no end caps; what is the change in the inside diameter? b) what would be the effect of adding end plates on the inside diameter?
E = $30 \times 10_6$ μ = .3

From (22)

a)

$$\sigma_t = p_i\left(\frac{1+(a/b)^2}{1-(a/b)^2}\right)$$




$$\sigma_t = 12,000\left(\frac{1+(.75/1.5)^2}{1-(.75/1.5)^2}\right)$$

$$\sigma_t = 20,000 psi$$

$$\sigma_r = -P_i = -12,000 psi$$

$$\sigma_L = 0$$

From (25)

$$\Delta D/D = \frac{\sigma_t - \mu(\sigma_r + \sigma_L)}{E}$$

$$\Delta D/D = \frac{20,000 - .3(-12,000+0)}{30 \times 10^6}$$

$$\Delta D/D = .000786$$

$$\Delta D = .000786(1.5) = .001179 in$$

Drom (24)
b)

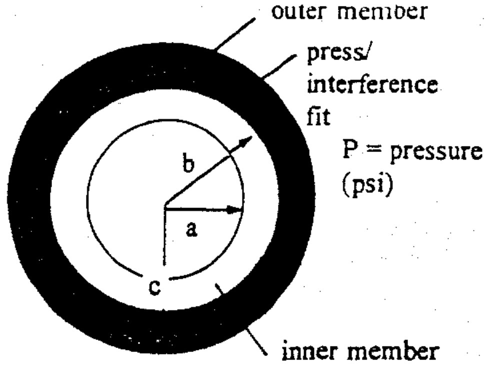
$$\sigma_L = \frac{P_i a^2}{(b+a)t} = \frac{12,000(.75)^2}{(1.5+.75)(.75)} = 4000 psi$$

$$\Delta D/D = \frac{20,000 - .3(-12,000+4000)}{30 \times 10^6} = .00746$$

$$\Delta D = .00746(1.5) = .001119 in$$

## INTERFERENCE FITS (PRESS FITS)

Interference fits create contact pressure at the interface allowing transmission of axial loads or torque between members.




$\delta$ = total deformation or radial interference (in)
$\delta_o$ = increase in radius of hole of outer cylinder
$\delta_i$ = decrease in radius of inner cylinder

$$\delta_o = \frac{bP}{E_o}\left(\frac{c^2+B^2}{c^2-b^2}+\mu_o\right)$$

$$\delta_i = -\frac{bP}{Ei}\left(\frac{B^2+a^2}{b^2-a^2}-\mu_i\right)$$

P = pressure (psi)
$E_o$ = modulus of outside material
$E_i$ = Poisson's Ratio outside material
$\mu_i$ = Poisson's Ratio inside material

$\delta$ = total deformation / radial interference

$$\delta = \delta_o - \delta_i = \frac{bP}{E_o}\left(\frac{c^2+b^2}{c^2-b^2}+\mu_o\right) + \frac{bP}{E_i}\left(\frac{b^2+a^2}{b^2-a^2}-\mu_i\right)$$

(26) $$p = \frac{\delta}{b/E_o\left(\frac{C^2=B^2}{C^2-B^2}+\mu_o\right)+b/E_i\left(\frac{b^2+a^2}{b^2-a^2}-\mu_i\right)}$$

when $E_o = E_i$ and $\mu_o = \mu_i$ (same material)

(27) $$P = \frac{E\delta}{b}\left(\frac{c^2-b^2)\,(b^2-a^2)}{2b^2(c^2-a^2)}\right)$$

Radial stress: $\delta_r = -P$ in each member at the contactin surfaces

Tangential or Hoop Stress:

- outer surface of the inner member

(28) $$\sigma_t = -P\left(\frac{b^2 + a^2}{b^2 - a^2}\right)$$

- INNER SURFACE OF THE OUTER MEMBER

(29) $$\sigma_t = P\left(\frac{c^2 + b^2}{c^2 - b^2}\right)$$

The axial load required to engage the respective parts or the axial load carrying capability is expressed by:

(30) $F = 2\,\pi\, b\, l\, P\, f$ *(lbs)*

where: l = length of interface
f = coefficient of friction at interface

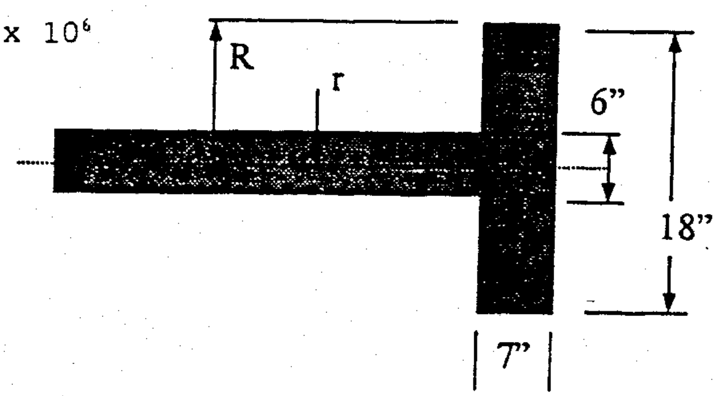
Torque transmitted:

(31) $T = 2\,\pi\, b_2\, l\, P\, f$ *(in-lbs)*

Problem

A solid 6 inch diameter (OD) steel shaft is to have a press fit in a steel flywheel with an outside dia of 18 in. The maximum tangential stress in the flywheel must not exceed 17,000 psi.

$E_s = 30 \times 10^6$




Find: 1. Diametral interference required

2. Force required to assemble parts if the flywheel is 7 in. thick
3. Find torque the joint will transmit due to the press fit pressure.

from equation (29)

$$\sigma_t = P\left(\frac{R^2 + r^2}{R^2 - r^2}\right)$$

$$17{,}000 = P\left(\frac{9^2 + 3^2}{9^2 - 3^2}\right) = P(1.25)$$

$P = 13{,}600$ psi

from equation (27) assuming 1 = 0 (solid shaft)

$$\delta = \frac{Pr}{E}\left(\frac{2R^2 r^2}{r^2(R^2 - r^2)}\right)$$

$$\delta = \frac{13{,}600(3)}{30 X 10^6}\left(\frac{2(9)^2(3)^2}{3)^2((9)^2-(3)^2)}\right)$$

$\delta = (1360)(2.25)\,10^{-6}$

$\delta = 0.00306$ in

Diametral interference = 2 $\delta$ = 0.00612 in

Assembly force from (30)

$F = 2\,\pi\,b\,l\,P\,f$
$F = 2\,\pi\,(r)\,l\,P\,f$
$F = 2\,\pi\,(3)\,(7)\,(13{,}600)\,(0.15)$
F = 269,172 lb

coefficient of friction is assumed to be 0.15

Torque from equation (31)

$T = 2\,\pi\,R^2\,l\,P\,f$
$T = 2\,\pi\,(3)^2\,(7)\,(13{,}600)\,(0.15)$
T = 807,517 in-lbs

## Objective

This section gives you an example of a 2D vibration design. I hope you all read through this example, since the concept is simple and once it shows in the test, you will be able to solve the problem very quick.

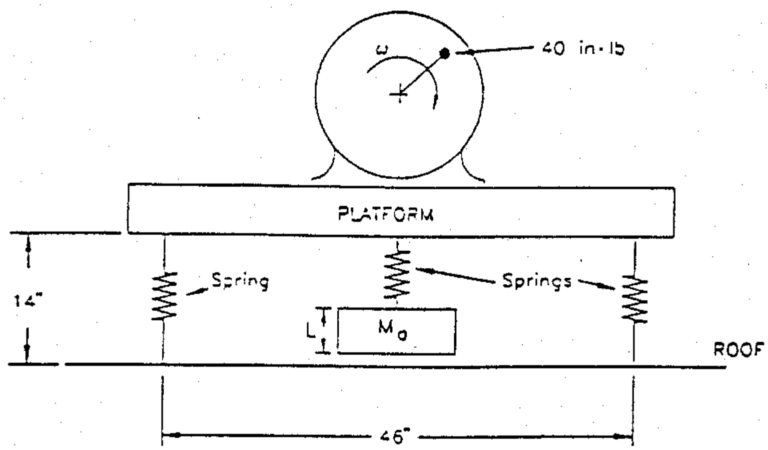
SITUATION:

A 2000-lb reciprocating compressor running at 200 rpm transmits a vertical force to its support equivalent to a rotating unbalance of 40 in-lb. The compressor is mounted on a platform which is supported by springs with a combined spring constant of 1500 lb/in. There is a clearance of 14 inches between the bottom of the platform and the flat roof of the building on which the compressor is mounted.

A vibration absorber consisting of a spring ($K_a$) and a mass ($M_a$) is to be hung under the compressor platform to minimize the vertical vibration of the platform.

REQUIREMENT:

Specify a spring constant ($K_a$) and the length (L) of a 6-inch diameter cylindrical steel rod of mass $M_a$ to be used for the vibration absorber.




NOT TO SCALE

SOLUTION:

Forcing frequency = $f_f$ = (200 rpm)($2\pi/60$) = 20.944 rad/sec.

Force generated = $mr(f_f)^2$ = $(40/386)(20.944)^2$ = 45.5 lb.

The vibration absorber must have a natural frequency equal to $f_f$ and must produce a force of 45.5 lb.

Therefore for the vibration absorber: 45.5 lb = $K_aX$ and $f_n$ = 20.944 rad/sec,
where: $K_a$ is the spring constant (a design factor),
X is the allowable displacement of the absorber, and
$f_n$ is the natural frequency of the absorber.

Since the space for the absorber is limited to 14 inches, selecting a length of the cylinder (L) to be approximately 6 inches, and a solid length of the spring to be 4 inches, leaves (14-6-4) = 4 inches for the total vibration. That is, 4 inches/2 or $\pm 2$ inches for the amplitude vibration of the absorber (X). Using a factor of safety of 2, 1 inch is selected as the amplitude of vibration of the absorber.

Since 45.5 lb = $K_aX$ and X = 1 inch, $\underline{K_a = 45.5 \text{ lb/inch}}$

$f_n = [K_a/M_a]^{1/2}$ where $M_a$ is the mass of the absorber.

Hence $M_a = K_a/(f_n)^2 = (45.5)/(20.944)^2 = 0.1037$

and the weight of the vibration absorber is (386)(0.1037) = 40 lb.

Since the weight is the product of the volume and the density of the steel rod,
40 lb = $\pi r^2 L\delta$ where the density of steel ($\delta$) is 0.28 lb/cubic inch and r = 3 inches.

Therefore L = $40/[9\pi(0.28)]$ = 5.1 inches

NOTE: Depending on the selected amplitude of vibration (X), the values of $K_a$ and L may change. However the solution procedure would not change. The selected amplitude of vibration must be consistent with the given constraints.

*damped* to some degree. If a free vibration is only slightly damped, its amplitude slowly decreases until, after a certain time, the motion comes to a stop. But damping may be large enough to prevent any true vibration; the system then slowly regains its original position (Sec. 19.8). A damped forced vibration is maintained as long as the periodic force which produces the vibration is applied. The amplitude of the vibration, however, is affected by the magnitude of the damping forces (Sec. 19.9).

## VIBRATIONS WITHOUT DAMPING

**19.2. Free Vibrations of Particles. Simple Harmonic Motion.** Consider a body of mass $m$ attached to a spring of constant $k$ (Fig. 19.1*a*). Since at the present time we are concerned only with the motion of its mass center, we shall refer to this body as a particle. When the particle is in static equilibrium, the forces acting on it are its weight **W** and the force **T** exerted by the spring, of magnitude $T = k\delta_{st}$, where $\delta_{st}$ denotes the elongation of the spring. We have, therefore,

$$W = k\delta_{st} \tag{19.1}$$

Suppose now that the particle is displaced through a distance $x_m$ from its equilibrium position and released with no initial velocity. If $x_m$ has been chosen smaller than $\delta_{st}$, the particle will move back and forth through its equilibrium position; a vibration of amplitude $x_m$ has been generated. Note that the vibration may also be produced by imparting a certain initial velocity to the particle when it is in its equilibrium position $x = 0$ or, more generally, by starting the particle from any given position $x = x_0$ with a given initial velocity $\mathbf{v}_0$.

To analyze the vibration, we shall consider the particle in a position $P$ at some arbitrary time $t$ (Fig. 19.1*b*). Denoting by $x$ the displacement $OP$ measured from the equilibrium position $O$ (positive downward), we note that the forces acting on the particle are its weight **W** and the force **T** exerted by the spring which, in this position, has a magnitude $T = k(\delta_{st} + x)$. Recalling (19.1), we find that the magnitude of the resultant **F** of the two forces (positive downward) is

$$F = W - k(\delta_{st} + x) = -kx \tag{19.2}$$

Thus the *resultant* of the forces exerted on the particle is proportional to the displacement *OP measured from the equilibrium position*. Recalling the sign convention, we note that **F** is always directed *toward* the equilibrium position $O$. Substituting for $F$ into the fundamental equation $F = ma$ and recalling that $a$ is the second derivative $\ddot{x}$ of $x$ with respect to $t$, we write

$$m\ddot{x} + kx = 0 \tag{19.3}$$

Note that the same sign convention should be used for the acceleration $\ddot{x}$ and for the displacement $x$, namely, positive downward.

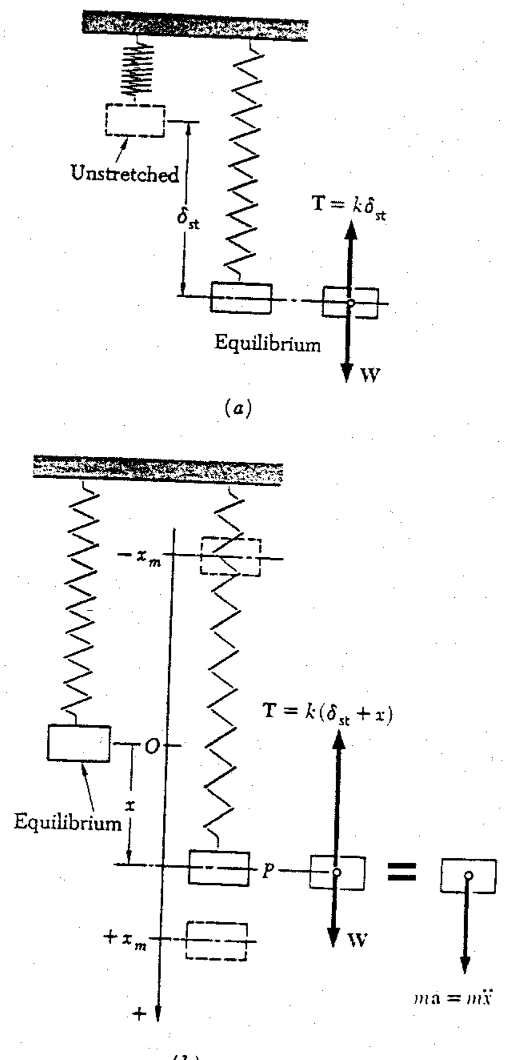
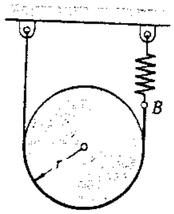
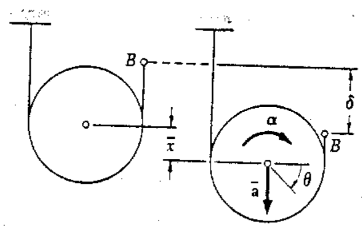


Fig. 19.1

## SAMPLE PROBLEM 19.2



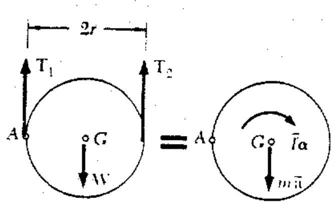

A cylinder of weight $W$ and radius $r$ is suspended from a looped cord as shown. One end of the cord is attached directly to a rigid support, while the other end is attached to a spring of constant $k$. Determine the period and frequency of vibration of the cylinder.




**Kinematics of Motion.** We express the linear displacement and the acceleration of the cylinder in terms of the angular displacement $\theta$. Choosing the positive sense clockwise and measuring the displacements from the equilibrium position, we write

$$\bar{x} = r\theta \qquad \delta = 2\bar{x} = 2r\theta$$

$$\alpha = \ddot{\theta} \circlearrowright \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \bar{\mathbf{a}} = r\ddot{\theta} \downarrow \tag{1}$$




**Equations of Motion.** The system of external forces acting on the cylinder consists of the weight $\mathbf{W}$ and of the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ exerted by the cord. We express that this system is equivalent to the system of effective forces represented by the vector $m\bar{\mathbf{a}}$ attached at $G$ and the couple $\bar{I}\boldsymbol{\alpha}$.

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \qquad Wr - T_2(2r) = m\bar{a}r + \bar{I}\alpha \tag{2}$$

When the cylinder is in its position of equilibrium, the tension in the cord is $T_0 = \frac{1}{2}W$. We note that for an angular displacement $\theta$, the magnitude of $\mathbf{T}_2$ is

$$T_2 = T_0 + k\delta = \tfrac{1}{2}W + k\delta = \tfrac{1}{2}W + k(2r\theta) \tag{3}$$

Substituting from (1) and (3) into (2), and recalling that $\bar{I} = \frac{1}{2}mr^2$, we write

$$Wr - (\tfrac{1}{2}W + 2kr\theta)(2r) = m(r\ddot{\theta})r + \tfrac{1}{2}mr^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

The motion is seen to be simple harmonic, and we have

$$p^2 = \frac{8}{3}\frac{k}{m} \qquad p = \sqrt{\frac{8}{3}\frac{k}{m}}$$

$$\tau = \frac{2\pi}{p} \qquad \tau = 2\pi\sqrt{\frac{3}{8}\frac{m}{k}} \blacktriangleleft$$

$$f = \frac{p}{2\pi} \qquad f = \frac{1}{2\pi}\sqrt{\frac{8}{3}\frac{k}{m}} \blacktriangleleft$$

# Objective

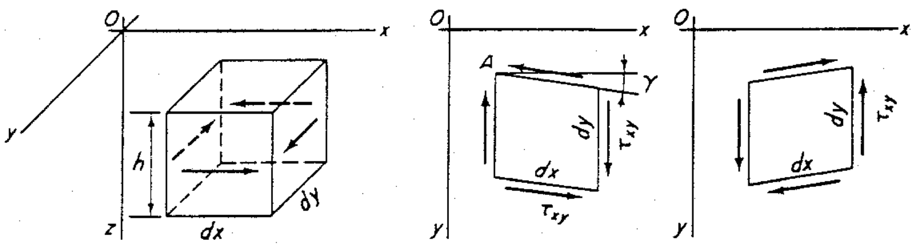
This concise description of Mohr's circle will help you understand its principle and applications. Please read through this section if you are still confusing, since the exam always has some sort of these questions.

As shown above, shallow ribs cause an increase in the bending stress. The deflection, however, has been decreased. If the ribs were made somewhat larger, the stress would be decreased, and the beam would be stronger.[4]

Ribbed structures are frequently made of a brittle material, such as cast iron, which is weak in tension. If possible, the ribs should be in compression. When a cast-iron body with parallel ribs is bent and the ribs are in tension, care must be exercised to make certain that all ribs are of the same height, or they may fail progressively beginning with the highest, and the full strength of the body could not be realized.

## *15. Shearing Stress*

Suppose an element is loaded by shearing stresses acting tangentially to its sides as shown in perspective in Fig. 1-19(a) or in plan in Fig. 1-19(b) and (c). Such loading causes no change in the length of the sides of the element, but merely produces a distortion or change in the value of the 90° angles in the corners.




*(a) Element loaded in shear* *(b) Positive shear* *(c) Negative shear*

**FIGURE 1-19 Element loaded by shearing or tangential stress.**

Shearing stresses are usually denoted by double subscripts. The first subscript indicates the direction of the normal to the plane under consideration, and the second subscript indicates the direction of the stress. Hence stress $\tau_{xy}$ lies in a plane whose normal is in the $x$-direction, while the stress acts in the $y$-direction. For similar reasons $\tau_{yx}$ indicates that the stress is in a plane perpendicular to the $y$-axis, and is parallel to the $x$-axis. Since the element is in equilibrium, the moments of the forces about a point, say $A$,

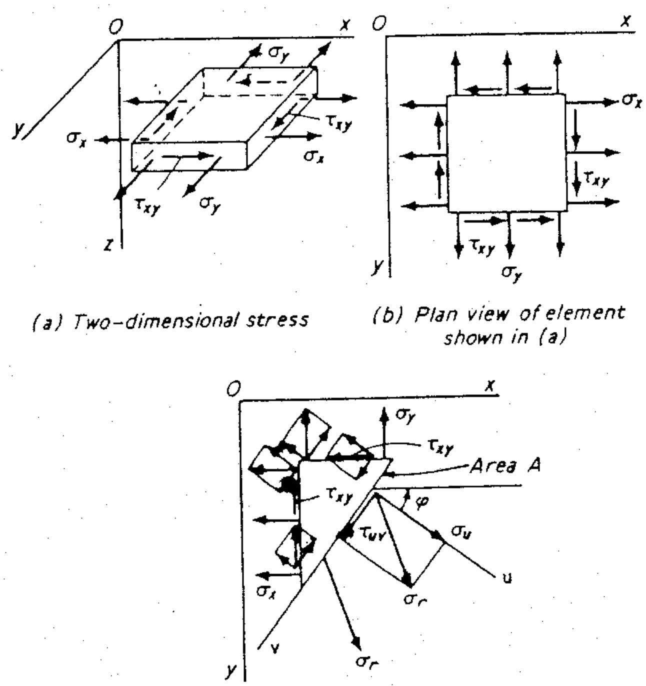
[4]See Marin, J., "Stiffness of Ribbed Plates," *Machine Design*, **19**, May 1947, p. 145; and Radich, E. A., "Strength and Stiffness of Ribbed Plates," *Machine Design*, **21**, Sept. 1949, p. 149.

## 19. Stresses in Any Given Direction

The stresses in a body, as found by the equations of this chapter, have definite directions. It is sometimes necessary to have the stresses at directions other than those given by the equations.

Figure 1-30(a) shows an element of a plate with the vertical surfaces subjected to the general two-dimensional state of stress. The element has been cut from a larger plate so that stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ represent the effect of the surrounding material on the element. A plan view of the element is shown in Fig. 1-30(b). Suppose stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ are known, and that it is necessary to find the values of the stresses on an inclined surface whose normal makes an angle $\varphi$ with the $x$-axis as shown in Fig. 1-30(c). Angle $\varphi$ is an arbitrarily chosen angle and determines the directions of the $u$- and $v$-axes.

Assume that stress $\sigma_r$ must be applied to the cut surface in order to maintain equilibrium of the remaining portion of the plate. Resultant stress $\sigma_r$ can be resolved into the components of normal stress $\sigma_u$ and shear stress $\tau_{uv}$ as shown.




(a) Two-dimensional stress

(b) Plan view of element shown in (a)

(c) Components of stress in directions u and v

**FIGURE 1-30** Shear and normal stress on element at any angle $\varphi$.

If the area of the inclined surface is $A$, then the area of the horizontal side of the body will be $A \sin \varphi$, and the area of the vertical side, $A \cos \varphi$. Since the plate of Fig. 1-30(c) is in equilibrium, the projections of the forces on the perpendicular to the inclined surface must be in equilibrium. Multiplication of stress by area and then by the appropriate trigonometric function gives the following equation for $\sigma_u$.

$$\sigma_u = 2\tau_{xy} \sin \varphi \cos \varphi + \sigma_x \cos^2 \varphi + \sigma_y \sin^2 \varphi$$

The trigonometric terms should be changed by the substitution of the equations involving the double angles. Then,

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\varphi + \tau_{xy} \sin 2\varphi \tag{37}$$

If the element in Fig. 1-30 is cut at 90° to the direction in sketch (c), summation of the forces will give the equation for the normal stress in the $v$ direction.

$$\sigma_v = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\varphi - \tau_{xy} \sin 2\varphi \tag{38}$$

Thus the normal stresses in the material at any desired angle $\varphi$ can be found by use of the above equations. Should the equation give a negative result, the corresponding stress is compressive.

In a similar manner, $\tau_{uv}$ can be found by making the sum of the projections of all forces parallel to the cut surface equal to zero. Hence,

$$\tau_{uv} = \tau_{xy}(\cos^2 \varphi - \sin^2 \varphi) - (\sigma_x - \sigma_y) \sin \varphi \cos \varphi$$

or

$$\tau_{uv} = \tau_{xy} \cos 2\varphi - \frac{\sigma_x - \sigma_y}{2} \sin 2\varphi \tag{39}$$

The shear stress $\tau_{uv}$ at any desired angle $\varphi$ can thus be found by Eq. (39). A positive result for $\tau_{uv}$ means that the stress is directed as in Fig. 1-30(c), and a negative result means that the stress is directed oppositely.

Angle $\varphi$ is positive when taken clockwise from the $x$-axis.

## 20. *The Mohr Circle*

A graphical solution to the combined stress problem, known as the Mohr circle, will now be given. Use of this method rather than the previously derived equations usually effects a considerable saving in time. However, certain conventions regarding signs and directions must be understood and carefully followed.

Figure 1-31 shows the perpendicular axes $\sigma$ and $\tau$. Normal stresses, regardless of the inclination of the surface on which they act, are plotted horizontally—positive, or tension, to the right of the origin, and negative,

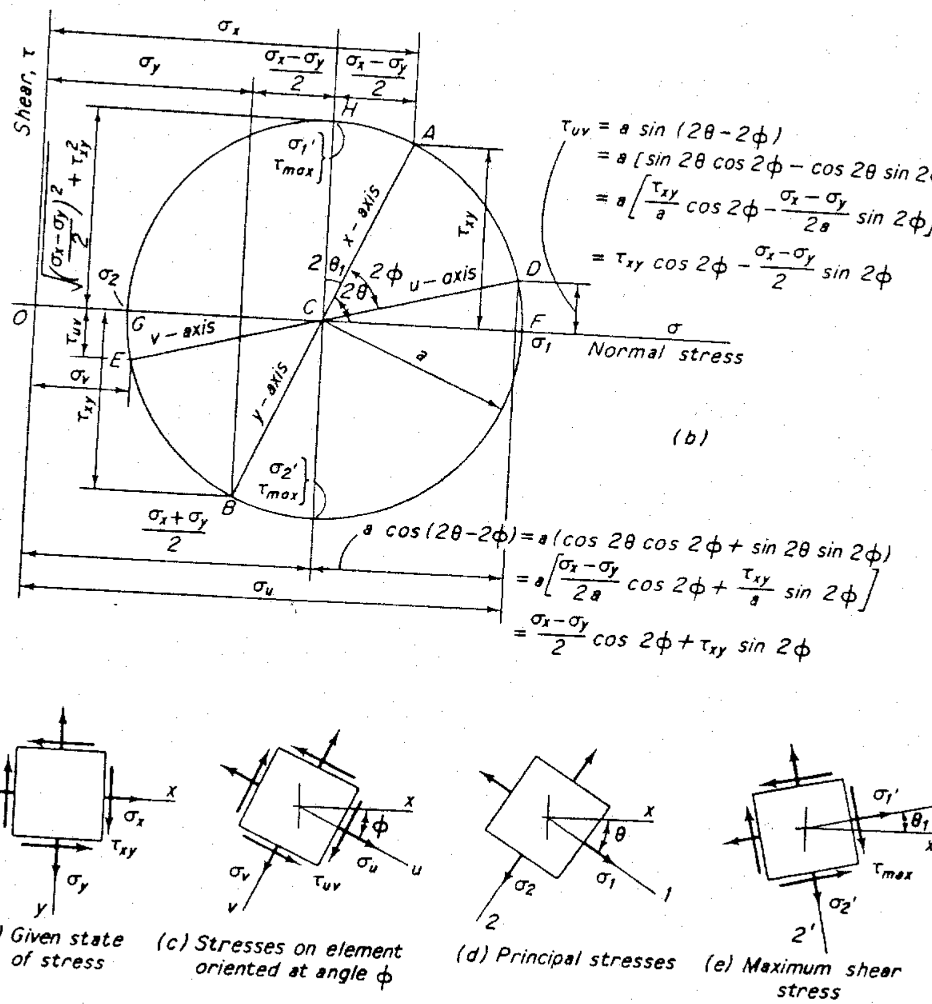



**FIGURE 1-31** Mohr circle for two-dimensional stress.

or compression, to the left. Shear stresses are plotted vertically upward or downward on the diagram. The normal and shear stresses at a point in the body thus become the coordinates of a point on the circle.

Stresses $\sigma_x$ and $\tau_{xy}$ acting on the right and left edges of the plate in Fig. 1-30(b) locate point $A$ in Fig. 1-31. Tension $\sigma_x$ is plotted to the right in accordance with the above-mentioned rule. Since shear stress $\tau_{xy}$ tends to rotate the element in a clockwise direction, it is plotted upward. Stresses $\sigma_y$ and $\tau_{xy}$ of the upper and lower edges of the plate shown in Fig. 1-30(b) locate point $B$ in Fig. 1-31. Tension $\sigma_y$ is plotted to the right. Since shear stress $\tau_{xy}$ on these surfaces tends to produce counterclockwise rotation, it is plotted

downward. The Mohr circle is drawn with line $AB$ as a diameter. Greater facility in the determination of angles will be obtained if radii $AC$ and $BC$ are marked $x$-axis and $y$-axis, respectively.

To find the stresses on an element oriented at angle $\varphi$, as shown in Fig. 1-31(c), the angle $2\varphi$ is laid off from $CA$ in the same direction as angle $\varphi$ is turned in the body. Diameter $DE$ is thus located.

The horizontal projection of $CD$ has the value shown in the figure. When this is added to $OC$, the result is the value of $\sigma_u$ as given by Eq. (37). The vertical projection of $CD$ has the value shown on the figure. This is equal to $\tau_{uv}$ as given by Eq. (39). It is plain that the coordinates of point $D$ of the circle are equal to the normal and shear stresses as found by the combined stress equations.

Stresses $\sigma_v$ and $\tau_{uv}$ for the surface, in Fig. 1-31(c), whose normal lies at angle $(90° + \varphi)$ from the $x$-axis, are given by the coordinates of point $E$.

A clockwise angle $\varphi$ on the body corresponds to a clockwise angle of $2\varphi$ on the circle, and vice versa.

Values of stresses $\sigma_u, \sigma_v$, and $\tau_{uv}$ change as angle $\varphi$ is changed. The maximum and minimum values of the normal stresses are called the principal stresses, and are designated $\sigma_1$ and $\sigma_2$, respectively. Their values can be found from the abscissas for points $F$ and $G$ in Fig. 1-31(b). The element for the principal stresses is oriented at angle $\theta$ to the $x$-axis as shown in Fig. 1-31(d). As shown by the circle, the value of $\theta$ can be found by the following equation.

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \qquad \text{for principal stresses} \tag{40}$$

The radius of the circle has the value shown. The equations for $\sigma_1$ and $\sigma_2$ are as follows.

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{41}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{42}$$

It should be noted that the sides of the element for principal stresses are free from shearing stress. If shear stress $\tau_{xy}$ should be equal to zero, stresses $\sigma_x$ and $\sigma_y$ would become the principal stresses, $\sigma_1$ and $\sigma_2$.[12]

The maximum shearing stress to which the material is subjected has a value equal to the radius of the circle. On the circle, point $H$ is located 90° from points $F$ and $G$ for principal stresses. In the body, the surfaces for maximum shear stress are thus inclined 45° to the surfaces for the principal stresses. The element of maximum shearing stress, as shown in Fig. 1-31(e),

[12]When the stress consists only of simple tension, it can be designated $\sigma$ rather than $\sigma_1$.

is inclined at $\theta_1$ to the $x$-axis. As shown by the circle, the value of $\theta_1$ can be found by the following equation.

$$\tan 2\theta_1 = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}, \qquad \text{for maximum shear stress} \tag{43}$$

The value of the maximum shearing stress is

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{44}$$

The circle of Fig. 1-31 indicates that at points of maximum shear, such as at $H$, normal stresses $\sigma_{1'}$ and $\sigma_{2'}$ are present whose value is given by the equation

$$\sigma_{1'} = \sigma_{2'} = \frac{\sigma_x + \sigma_y}{2} \tag{45}$$

When shear stress $\tau$ is equal to zero, the radius of the circle, or the maximum shearing stress is equal to

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) \tag{46}$$

*Example 15.* Let the state of stress at some point in a body be defined as follows.

$$\sigma_x = 20{,}000 \text{ psi}, \qquad \sigma_y = -4{,}000 \text{ psi}, \qquad \tau_{xy} = 5{,}000 \text{ psi}$$

(a) Draw the view of the element for the given state of stress and mark values thereon.

(b) Draw the Mohr circle for the given state of stress and mark completely.

(c) Draw the element oriented 30° clockwise from the $x$-axis and show values of all stresses.

(d) Draw the element correctly oriented for principal stresses and show values.

(e) Draw the element for maximum shear stress and mark values of all stresses.

*Solution.* (a), (b). The given state of stress and the Mohr circle are shown in Fig. 1-32(a) and (b), respectively.

(c) Diameter $ECD$ should be drawn at 60° clockwise to the $x$-axis of the circle, and stresses $\sigma_u$ and $\sigma_v$ scaled and placed on the element of Fig. 1-32(c). Since point $D$ lies below the $\sigma$-axis, shear stress $\tau_{uv}$ crosses the $u$-axis of Fig. 1-32(c) in the direction that causes a counterclockwise moment on the element. Likewise, since $E$ lies above the $\sigma$-axis, stress $\tau_{uv}$ crosses the $v$-axis in sketch (c) in the direction that causes a clockwise moment on the element.

(d) Principal stresses $\sigma_1$ and $\sigma_2$, together with their angle of inclination, are scaled directly from the circle, and are shown acting on an element properly oriented in Fig. 1-32(d).

(e) The maximum shear stress $\tau_{max}$ and the corresponding normal stress $\sigma_{1'}$ are shown on the element of Fig. 1-32(e). The arrows are directed in accordance with the previously explained rules.

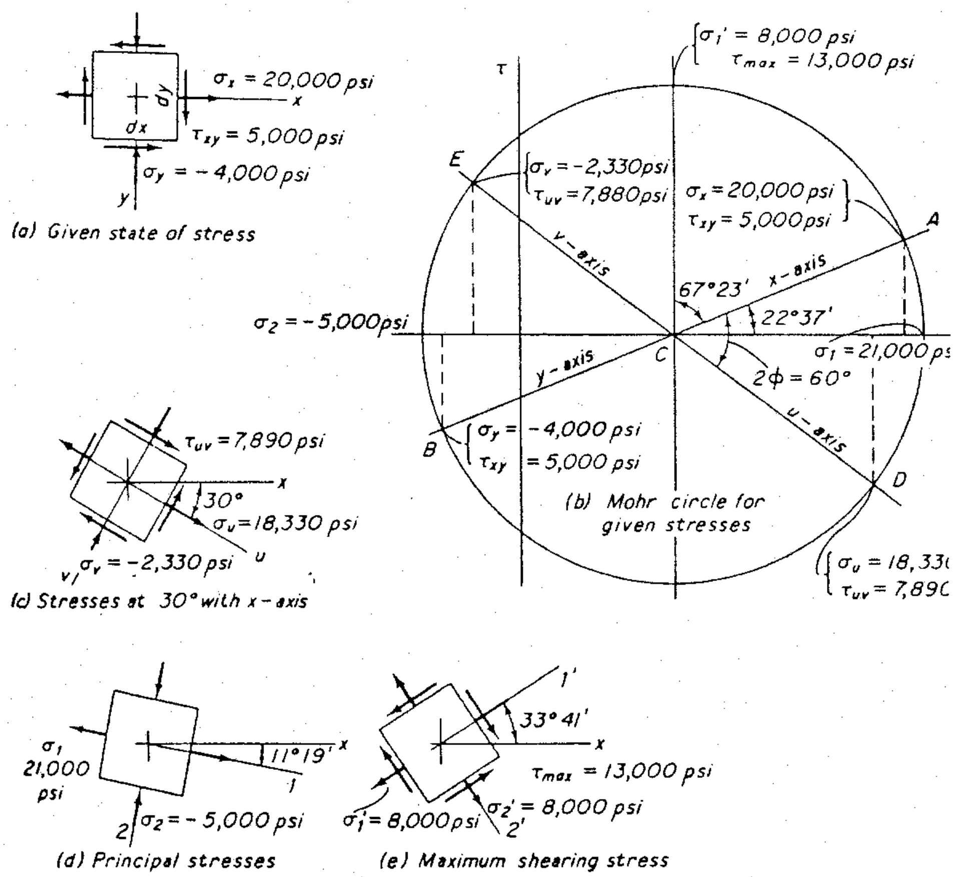



**FIGURE 1-32** Solution of Example 15 by Mohr circle.

The advantages of the graphical method for solving combined stress problems should now be apparent. Not only is the method more rapid, but the state of stress for any direction can be scaled directly from the circle. When the equations are used for solving a problem, a separate computation must be made for each desired direction. The Mohr circle also aids in forming a mental picture of the state of stress in the body. In working problems, care must be exercised that all necessary information is placed on the drawing for the circle as well as on the views of the various elements.

The reader should check all values shown in Fig. 1-32(c), (d), and (e) by using the appropriate equations. Note that for $\varphi = 30°$, Eq. (39) gives a negative result for $\tau_{uv}$. This result checks with the circle and indicates that the shear stress for this direction is acting oppositely to that shown in Fig. 1-30(c).

# Objective

This chapter gives you an example of fluctuating load analysis. You will be able to know how to determine the factor of safety in this case.

## 12. *Ductile Materials with Combined Steady and Alternating Stress. Modified Goodman Diagram*

In many strength problems, the major components of stress are static, with less accurately known alternating stresses superposed. Most failures originate with stresses of this type. The problem presents great difficulties because of the fundamentally different mechanisms of failure in the two sources of stress.

Suppose the tensile load $P$ on the bar of Fig. 2-25(a) is continuously varying in magnitude. This load can be considered as being made up of two parts, the steady or average load $P_{av}$, and the variable or range load $P_r$. The maximum load is equal to the average load plus the range load; the minimum load is equal to the average minus the range load. Normal stresses $\sigma_{av}$ and $\sigma_r$ are found by dividing loads $P_{av}$ and $P_r$ by the cross-sectional area $A$. When the average stress is high, the material will safely carry only a small additional range component. However, if the average stress is small, a larger range component can be permitted.

To take care of the unlimited number of combinations of range and average stress, the line of failure of the material must be used. Specimens are tested with fluctuating loads that are low enough to permit continuous operation but high enough so that any increase in either the average or range load will eventually cause failure.

The results are plotted, $\sigma_{av}$ horizontally and $\sigma_r$ vertically, for each specimen tested. A sufficient number of test points gives the curve of failure for the material. It is a broad sweeping curve running from $A$ to $B$ in Fig. 2-25. For a static stress, failure occurs at the ultimate or point $B$ in the figure. For a completely reversing stress (zero average), failure occurs at the endurance

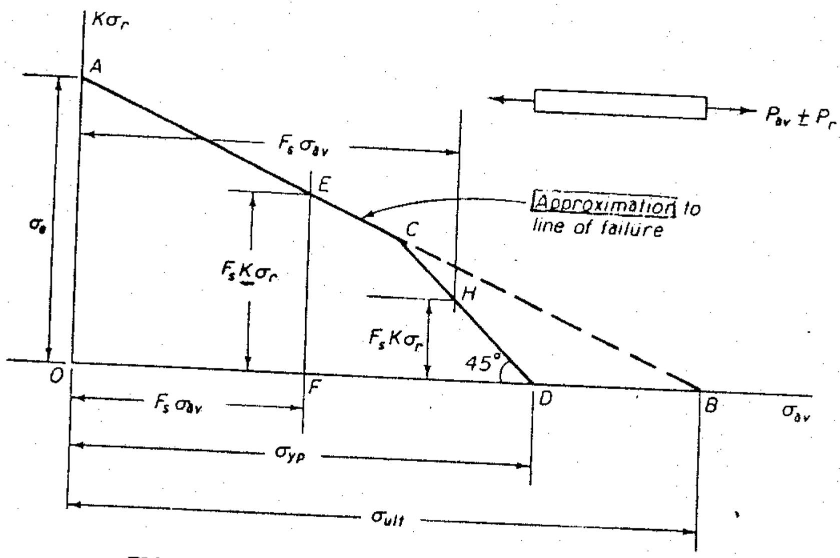



**FIGURE 2-25** **Working stress diagram for fluctuating load.**

limit or point $A$ in Fig. 2-25. For combinations of average and range stress, failure occurs at intermediate points between $A$ and $B$. Such experimentally determined curves in general are not available to the designer and an approximation must be used.

For this the curve $ACD$ as drawn in the figure can be used. All points as located by average and range stress lying on $ACD$ can be considered as equally safe if fatigue failure is not to occur. As mentioned previously, a stress concentration factor $K$, if present, will be applied only to the range component. Since a factor of safety is employed both components $\sigma_{av}$ and $K\sigma_r$ must be multiplied by $F_s$ to cause the resulting stress to fall on $ACD$, the assumed line of failure.

Thus the average and range components locating a point such as $E$ are considered as being equivalent to the static stress $OB$ or $\sigma_{ult}$. An equation for the equivalent static stress $\sigma$ is easily derived. By similar triangles in Fig. 2-25:

$$\frac{FB}{EF} = \frac{OB}{OA} \quad \text{or} \quad FB = \frac{OB}{OA} \times EF = \frac{\sigma_{ult}}{\sigma_e} F_s K\sigma_r$$

Equivalent static stress: $OB = OF + FB$

$$\sigma_{ult} = F_s\sigma_{av} + F_s K \frac{\sigma_{ult}}{\sigma_e}\sigma_r$$

$$\sigma = \frac{\sigma_{ult}}{F_s} = \sigma_{av} + K\frac{\sigma_{ult}}{\sigma_e}\sigma_r \tag{11}$$

or

$$\frac{\sigma_{av}}{\sigma_{ult}} + \frac{K\sigma_r}{\sigma_e} = \frac{1}{F_s} \tag{11a}$$

*Example 7.* A part with a machined surface has continuously varying tension loads $P_{max} = 45{,}000$ lb and $P_{min} = 15{,}000$ lb. Material tests $\sigma_{ult} = 90{,}000$ psi, and $\sigma_{yp} = 70{,}000$ psi. A stress concentration factor of 1.42 is present. Area of the part is 2.5 in.$^2$ Find the factor of safety.

*Solution.* $$P_{av} = \frac{45{,}000 + 15{,}000}{2} = 30{,}000 \text{ lb}, \qquad \sigma_{av} = \frac{30{,}000}{2.5} = 12{,}000 \text{ psi}$$

$$P_r = \frac{45{,}000 - 15{,}000}{2} = 15{,}000 \text{ lb}, \qquad \sigma_r = \frac{15{,}000}{2.5} = 6{,}000 \text{ psi}$$

By Fig. 2-11: $\sigma_e = 34{,}000$ psi

A rough sketch indicates that the working stress point $E$ will lie on the upper limb of the curve in Fig. 2-25.

By Eq. (11a): $$\frac{12{,}000}{90{,}000} + \frac{1.42 \times 6{,}000}{34{,}000} = \frac{1}{F_s}$$

$$0.1333 + 0.2506 = \frac{1}{F_s}$$

$$F_s = 2.60$$

*Example 7A.* A part with a machined surface has continuously varying tensile loads $P_{max} = 200{,}000$ N and $P_{min} = 68{,}000$ N. Material tests $\sigma_{ult} = 600$ MPa. A stress concentration factor of 1.42 is present. Area of the part is 1,600 mm$^2$. Find the factor of safety for the part

*Solution.*

$$P_{av} = \frac{200{,}000 + 68{,}000}{2} = 134{,}000 \text{ N} \qquad \sigma_{av} = \frac{134{,}000}{1{,}600} = 83.75 \text{ MPa}$$

$$P_r = \frac{200{,}000 - 68{,}000}{2} = 66{,}000 \text{ N} \qquad \sigma_r = \frac{66{,}000}{1{,}600} = 41.25 \text{ MPa}$$

$$\sigma_{ult} = 145 \times 600 = 87{,}000 \text{ psi}$$

By Fig. 2-11: $\sigma_e = 34{,}000$ psi $= 234$ MPa

By Eq. (11a): $$\frac{83.75}{600} + \frac{1.42 \times 41.25}{234} = \frac{1}{F_s}$$

$$0.1396 + 0.2503 = \frac{1}{F_s}$$

$$F_s = 2.56$$

When the average stress is greater the loading point may lie on the lower limb of the curve as at $H$ in Fig. 2-25. Here, the equivalent static tension $\sigma$ is equal to $OD$ or $\sigma_{yp}$. Then

$$F_s(\sigma_{av} + K\sigma_r) = \sigma_{yp} \tag{12}$$

*Example 8.* Same data as for Example 7 except $P_{max} = 55{,}400$ lb and $P_{min} = 44{,}600$ lb. Find the $F_s$.

*Solution.* A sketch will indicate that the loading point will lie on the lower limb of the curve in Fig. 2-25.

$$P_{av} = \frac{55{,}400 + 44{,}600}{2} = 50{,}000 \text{ lb} \qquad \sigma_{av} = \frac{50{,}000}{2.5} = 20{,}000 \text{ psi}$$

$$P_r = \frac{55{,}400 - 44{,}600}{2} = 5{,}400 \text{ lb} \qquad \sigma_r = \frac{5{,}400}{2.5} = 2{,}160 \text{ psi}$$

By Eq. (12): $$F_s(20{,}000 + 1.42 \times 2{,}160) = 70{,}000$$

$$F_s = \frac{70{,}000}{23{,}067} = 3.03$$

## *13. Sensitivity to Stress Concentration*

The actual reduction in fatigue strength, as indicated by the foregoing theoretical stress concentration factors, is approached only by large parts made of fine-grained heat-treated steel. The effect of stress concentration in coarse-grained annealed steels may be considerably less. Small specimens are affected less by stress concentration than larger parts made of the same material. The size effect in steel is attributed mainly to the grain size of the material. When the crystal size is taken into account, it is seen that there is not complete geometric similarity between large and small specimens of the same material. Although a heat-treated part of expensive alloy steel may have a higher endurance limit than one made of a softer nonheat-treated material, the advantage may be largely lost in the presence of a stress concentration.

A wide variation exists in the *notch sensitivity* of different materials. For some quenched and tempered steels the effect of a sharp notch may be so great that a high-strength material may be no better in fatigue than one of lower strength. Materials that work harden rapidly, such as the 18-8 stainless steels, may show great resistance to loss of fatigue strength due to notches. Notches have but little effect on the fatigue strength of cast iron, but may have a large effect as far as impact loads are concerned. However, the impact strength of some steels is but little affected by notches.

Methods are available for making a quantitative estimate of the sensitivity of a steel to stress concentration,[6] but the methods are beyond the scope of this book. However, when the full theoretical stress concentration factor is applied to the fluctuating component, the result will usually be on the safe side.

[6]See Peterson, R. E., "Relation between Life Testing and Conventional Tests of Materials," *ASTM Bull.*, **133**, Mar. 9, 1945. See also p. 8 of Reference 9, Chapter 4 of Reference 8, and Chapter 13 of Reference 11, end of chapter.

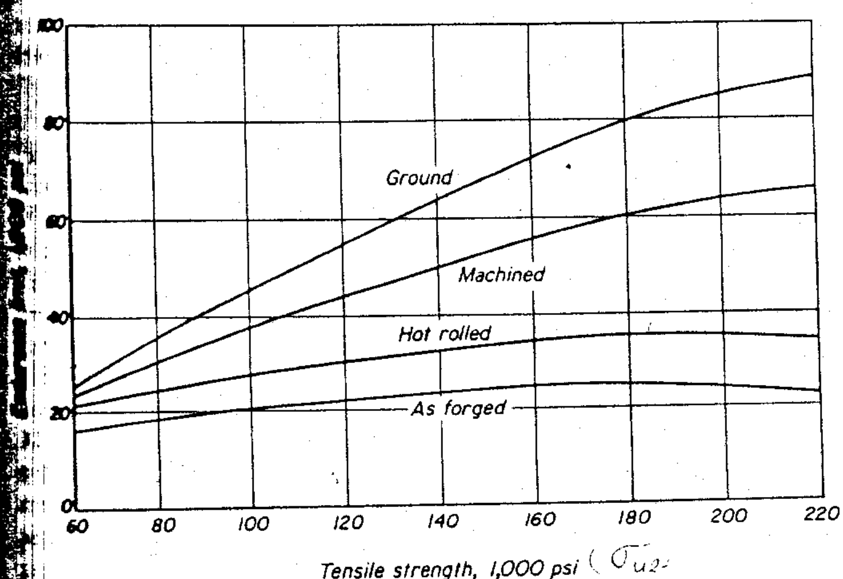



FIGURE 2-11 Relation between endurance limit and tensile strength for unnotched specimens in reversed bending.

give the most consistent results with respect to fatigue strength. For steel castings and cast iron, the endurance limit is about 40 per cent of the ultimate strength. Apparently no relationship exists between the endurance limit and the yield point, impact strength, or ductility. Experiments have shown that the endurance limit for reversed torsion is about 0.56 of that for reversed bending.[3]

Fatigue cracks can start not only at easily recognized changes of form, but also at frequently overlooked stress raisers, such as file and tool marks, accidental and grinding scratches, quenching cracks, or part number and inspection stamps, which produce a high value for the stress and serve as the starting point for the progressive failure. The attention of the designer must therefore be focused on such "sore spots" whenever they are located in a region of high tension stress.

Since fatigue cracks are due to tensile stress, a residual stress of tension on the surface of the part constitutes an additional fatigue hazard. Such a tensile stress, for example, may arise from a cold-working operation on the part without stress relieving. Parts that are finished by grinding frequently have an extremely thin surface layer, which is highly stressed in tension. Such residual stresses, combined with the tensile stress from the loading, may give a resultant stress sufficiently great to cause a fatigue crack to start.

[3]See Fig. 7, p. 7, of Reference 9, end of chapter.

# Objective

This chapter gives you an example for a static welding design, including how to determine the max. shearing stress and factor of safety.

## 4. Eccentrically Loaded Welds—Static Loads

When the load on a welded joint is applied eccentrically, the effect of the torque or moment must be taken into account as well as the direct load. The state of stress in such a joint is complicated, and it is necessary to make simplifying assumptions.

When a joint consists of a number of welds, it is customary to assume that the moment stress at any point is proportional to the distance from the

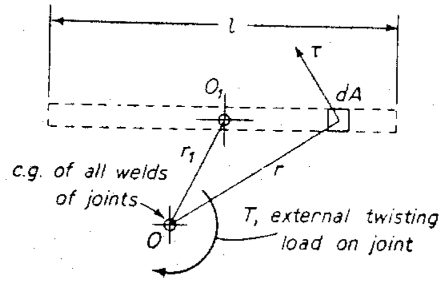



**FIGURE 7-4 Stress on element of eccentrically loaded welded joint.**

center of gravity of the group of welds. Let the weld shown in Fig. 7-4 be one of a group forming a joint with the center of gravity of all the weld areas at $O$. The moment stress $\tau$ acts perpendicularly to radius $r$ on element $dA$ of the weld. The external moment or torque $T$ is equal to the moment from stress $\tau$ integrated over all the welds of the joint.

$$T = \int r\,\tau\, dA = \int \frac{\tau}{r} r^2\, dA = \frac{\tau}{r}\int r^2\, dA = \frac{\tau J}{r}$$

$$\tau = \frac{Tr}{J} \tag{3}$$

Ratio $\tau/r$ is a constant since the stress is assumed to vary directly with $r$. The integral $\int r^2\, dA$ in Eq. (3) has been replaced by $J$, the polar moment of inertia about $O$ for the group of welds. For the maximum torsional stress, the value of $r$ to the point furthest removed from the center of gravity $O$ must be used. The stress from the direct load must be added vectorially to the moment stress in order to obtain the resultant stress. For static loads, it is usual practice to assume that the direct stress in a weld is uniformly distributed throughout its area.

The parallel axis equation can be used for finding the value for $J$ for a weld about an axis through $O$ perpendicular to the plane of the weld. For the weld in Fig. 7-4 this equation would be written

$$J = J_o + Ar_1^2 \tag{4}$$

The area $A$ in this equation refers to the throat area of the weld. The fact that the throat for a fillet weld is inclined at 45° to the plane of the joint has no effect on the value of $J$. Radius $r_1$ extends from the center $O_1$ of the weld to the center of gravity $O$ for the group. Symbol $J_o$ represents the moment of inertia of the single-weld area about its own center $O_1$. This value can be found from the following equation.

$$J_o = \frac{Al^2}{12} \tag{5}$$

where $A$ is again the throat area and $l$ is the length of the weld.

When Eq. (5) is substituted into Eq. (4), the result is

$$J = A\left(\frac{l^2}{12} + r_1^2\right) \tag{6}$$

The value of $J$ for each weld about $O$ should be computed by Eq. (6); the results are added to obtain the moment of inertia of the entire joint.

*Example 2.* An eccentrically loaded bracket is welded to its support as shown in Fig. 7-5. If the load is steady, find the value of the maximum stress in the weld. Find the factor of safety if the yield strength of the weld metal is 50,000 psi.

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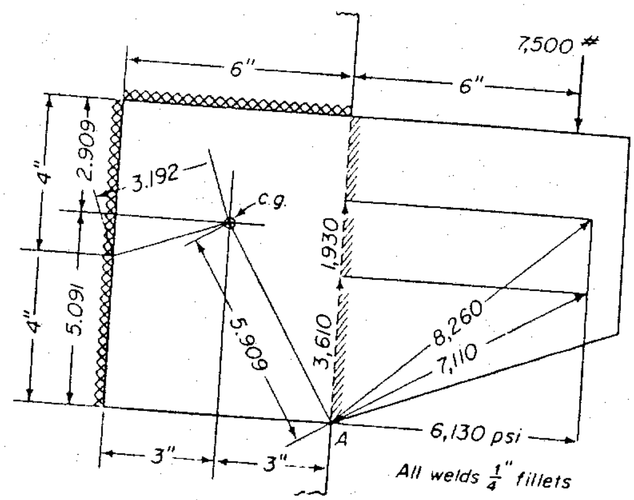


FIGURE 7-5 **Example 2.**

*Solution.* Consider the bracket as the free body. By symmetry the center of gravity of the welds in the $x$-direction is halfway between the vertical welds. The center of gravity in the vertical direction is found by taking moments about the top weld.

$$22\bar{y} = 2 \times 8 \times 4$$

$$\bar{y} = 2.909 \text{ in. from top}$$

Width of throat: $b = 0.707h = 0.707 \times \frac{1}{4} = 0.1768$ in.

For vertical weld: $A = 0.177 \times 8 = 1.414$ in.$^2$

$$r_1 = \sqrt{1.091^2 + 3^2} = \sqrt{10.1903} = 3.192 \text{ in.}$$

By Eq. (6): $J = 1.4142\left(\frac{8^2}{12} + 3.192^2\right) = 21.953$ in.$^4$

For top weld: $A = 0.1768 \times 6 = 1.061$ in.$^2$

By Eq. (6): $J = 1.061\left(\frac{6^2}{12} + 2.909^2\right) = 12.159$ in.$^4$

Total J: $J = 2 \times 21.953 + 12.159 = 56.065$ in.$^4$

Total area: $A = 2 \times 1.414 + 1.061 = 3.889$ in.$^2$

Direct stress: $\tau = \frac{7{,}500}{3.889} = 1{,}930$ psi

The maximum stress will occur at $A$.

Radius $OA$: $r = \sqrt{5.091^2 + 3^2} = 5.909$ in.

Torsional stress: $\tau = \frac{Tr}{J} = \frac{7{,}500 \times 9 \times 5.909}{56.065} = 7{,}110$ psi

This stress is directed perpendicular to the radius from the center of gr. It is now resolved into the components shown. The total vertical cor 3,610 + 1,930 or 5,540 psi.

Resultant stress: $\tau = \sqrt{5{,}540^2 + 6{,}130^2} = 8{,}260$ psi

By Eq. (2): $F_s = \dfrac{0.5\sigma_{yp}}{\tau} = \dfrac{25{,}000}{8{,}260} = 3.03$

The foregoing method assumes that failure would take place b) the weld throats. This is correct only when the weld pattern is a c about the center of gravity. The results for a pattern like Fig. 7-5 w approximate.

***Example 3.*** Find the value of static force $P$ in Fig. 7-6 if electrode E6 at a factor of safety equal to 2.

(a) All welds are $\frac{1}{4}$-in. fillets.

(b) Welds on the left side are $\frac{1}{4}$ in. and on the right side are $\frac{1}{2}$ in.

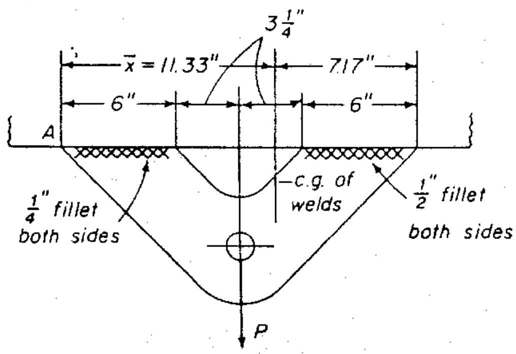


**FIGURE 7-6 Example 3.**

*Solution.* (a) By Table 7-3, for rod E6010, $\sigma_{yp} = 50{,}000$ psi.

Throat area of all welds: $A = 4 \times \frac{1}{4} \times 6 \times 0.707 = 4.242$ in.$^2$

Then: $P = \tau A = \dfrac{25{,}000}{2} \times 4.242 = 53{,}020$ lb

(b) Left, throat area: $A = 2 \times \frac{1}{4} \times 6 \times 0.707 = 2.121$ in.$^2$ (both welds

Right, throat area: $A = 2 \times \frac{1}{2} \times 6 \times 0.707 = 4.242$ in.$^2$ (both welds

Total area: $A = 6.363$ in.$^2$

Take moments at left end: $\bar{x} = \dfrac{2.121 \times 3 + 4.242 \times 15.5}{6.363}$

$= 11.33$ in.

For left welds: Eq. (6): $J = 2.121\left(\frac{6^2}{12} + 8.33^2\right) = 153.65 \text{ in.}^4$

For right welds: Eq. (6): $J = 4.242\left(\frac{6^2}{12} + 4.17^2\right) = 86.37 \text{ in.}^4$

Total: $J = 153.65 + 86.37 = 240.02 \text{ in.}^4$

Direct stress: $\tau = \frac{P}{6.363} = 0.1572P$

Moment arm of the load is 11.33 − 9.25 or 2.08 in.

Moment stress at $A$: $\tau = \frac{Tr}{J} = \frac{P2.08 \times 11.33}{240.02} = 0.0984P$

Total stress at $A$: $\tau = 0.1572P + 0.0984P = 0.2556P$

By Eq. (2): $0.2556P = \frac{25{,}000}{2}$ or $P = 48{,}900$ lb

It should be noted that, although the welds of part (b) are larger, because of the eccentricity, the carrying capacity of the joint has been actually reduced. It is generally advantageous to maintain symmetry in the design of a welded joint.

*Example 3A.* In Fig. 7-6, let the successive dimensions across the top be 150 mm, 82 mm, 82 mm, and 150 mm. The welds on the left are 6 mm and on the right are 12 mm. Find the value of the static load $P$ if electrode E6010 is used at a factor of safety of 2.

*Solution.* By Table 7-3, for rod E6010, $\sigma_{yp} = 50{,}000$ psi = 34.5 MPa.

Left, throat area: $A = 2 \times 6 \times 150 \times 0.707 = 1{,}273 \text{ mm}^2$ both welds

Right, throat area: $A = 2 \times 12 \times 150 \times 0.707 = 2{,}546 \text{ mm}^2$ both welds

Total area, $A = 3{,}819 \text{ mm}^2$

Moments at left: $\bar{x} = \frac{1{,}273 \times 75 + 2{,}546 \times 389}{3{,}819} = 284.3$ mm from left

Distance, c.g. to center: left $= 284.3 - 75 = 209.3$ mm

right $= 464 - (284.3 + 75) = 104.7$ mm

By Eq. (6), left welds: $J = 1{,}273\left(\frac{150^2}{12} + 209.3^2\right) = 58{,}152{,}000 \text{ mm}^4$

right welds: $J = 2{,}546\left(\frac{150^2}{12} + 104.7^2\right) = 32{,}683{,}000 \text{ mm}^4$

Total: $J = 58{,}152{,}500 + 32{,}683{,}000 = 90{,}835{,}000 \text{ mm}^4$

Direct stress: $\tau = \frac{P}{3{,}819} = 0.000\ 261\ 8P$

Moment arm of load $= 284.3 - (150 + 82) = 52.3$ mm

Torsion stress at $A$: $$\tau = \frac{Tr}{J} = \frac{52.3P284.3}{90{,}835{,}500} = 0.000\ 163\ 7P$$

Total stress at $A$: $$\tau = (0.000\ 261\ 8 + 0.000\ 163\ 7)P$$
$$= 0.000\ 425\ 5P$$

By Eq. (2): $$0.000\ 425\ 5P = \frac{17.25}{2} \quad \text{or} \quad P = 202{,}700\ \text{N}$$

The method explained in this section cannot be considered an exact analysis of weld stresses, but should be looked upon simply as a reasonable effort to take into account the fact that capacity of a joint to resist moment loads is increased by locating the welds further from the center. The theory also assumes that the weld stresses are within the elastic limit of the weld material, and that any effects from the deflection of the welded parts can be neglected.